

# Infinite-dimensional stochastic differential equations arising from Airy random point fields

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**Abstract:** We identify infinite-dimensional stochastic differential equations (ISDEs) describing the stochastic dynamics related to  $\text{Airy}_\beta$  random point fields with  $\beta = 1, 2, 4$ . We prove the existence of unique strong solutions of these ISDEs. When  $\beta = 2$ , this solution is equal to the stochastic dynamics defined by the space-time correlation functions obtained by Spohn and Johansson among others. We develop a new method to construct a unique, strong solution of ISDEs. We expect that our approach is valid for other soft-edge scaling limits of stochastic dynamics arising from the random matrix theory.

## 1 Introduction

Gaussian ensembles are introduced as random matrices with independent elements of Gaussian random variables, under the constraint that the joint distribution is invariant under conjugation with appropriate unitary matrices. The ensembles are divided into classes according to whether their elements are real, complex, or real quaternion, and their invariance by orthogonal (Gaussian orthogonal ensemble, GOE), unitary (Gaussian unitary ensemble, GUE) and unitary symplectic (Gaussian symplectic ensemble, GSE) conjugation.

The distribution of eigenvalues of the ensembles with size  $n \times n$  is given by

$$(1.1) \quad m_\beta^n(d\mathbf{x}_n) = \frac{1}{Z} \left\{ \prod_{i < j}^n |x_i - x_j|^\beta \right\} \exp \left\{ -\frac{\beta}{4} \sum_{k=1}^n |x_k|^2 \right\} d\mathbf{x}_n,$$

where  $\mathbf{x}_n = (x_1, \dots, x_n)$  and  $d\mathbf{x}_n = dx_1 \cdots dx_n$ . The GOE, GUE, and GSE correspond to  $\beta = 1, 2$  and  $4$ , respectively. The probability density coincides with the Boltzmann factor normalized by the partition function  $Z$  for a log-gas system at three specific values of the inverse temperature  $\beta = 1, 2$ , and  $4$ . The measures  $m_\beta^n$  still make sense for any  $0 < \beta < \infty$ , and are examples of log-gasses [6].

Let  $\mu_\beta^n$  be the distribution of  $n^{-1} \sum \delta_{x_i/\sqrt{n}}$  under  $m_\beta^n(d\mathbf{x}_n)$ . Wigner's celebrated semi-circle law states that the sequence  $\{\mu_\beta^n\}$  weakly converges to the nonrandom  $\sigma_{\text{semi}}(x)dx$  in the space of probability measures on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . Here  $\sigma_{\text{semi}}$  is defined as

$$\sigma_{\text{semi}}(x) = \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}_{(-2,2)}(x).$$

There exist two typical thermodynamic scalings in (1.1), namely bulk and soft edges. The former (centered at the origin) is given by the correspondence  $x \mapsto x/\sqrt{n}$ , which yields the random point field (RPF)  $\mu_{\text{sin},\beta}^n$  with labeled density  $m_{\text{sin},\beta}^n$  such that

$$(1.2) \quad m_{\text{sin},\beta}^n(d\mathbf{x}_n) = \frac{1}{Z} \left\{ \prod_{i < j}^n |x_i - x_j|^\beta \right\} \exp \left\{ -\frac{\beta}{4n} \sum_{k=1}^n |x_k|^2 \right\} d\mathbf{x}_n.$$

The latter, on the other hand, is centered at  $2\sqrt{n}$  given by the correspondence  $x \mapsto xn^{-1/6} + 2\sqrt{n}$  with density  $m_{\text{Ai},\beta}^n$  such that

$$(1.3) \quad m_{\text{Ai},\beta}^n(d\mathbf{x}_n) = \frac{1}{Z} \left\{ \prod_{i < j}^n |x_i - x_j|^\beta \right\} \exp \left\{ -\frac{\beta}{4} \sum_{k=1}^n |2\sqrt{n} + n^{-1/6}x_k|^2 \right\} d\mathbf{x}_n.$$

Suppose  $\beta = 2$ . The limit RPF  $\mu_{\text{sin},2}$  of the finite-particle system (1.2) is then the determinantal RPF with  $n$ -correlation functions  $\rho_{\text{sin},2}^n$  defined as

$$\rho_{\text{sin},2}^n(\mathbf{x}_n) = \det[K_{\text{sin},2}(x_i, x_j)]_{i,j=1}^n.$$

Here  $K_{\text{sin},2}$  is a continuous kernel such that, for  $x \neq y$ ,

$$K_{\text{sin},2}(x, y) = \frac{\sin\{\pi(x - y)\}}{\pi(x - y)}.$$

The limit RPF  $\mu_{\text{Ai},2}$  of the finite-particle system (1.3) is also the determinantal RPF with  $n$ -correlation functions  $\rho_{\text{Ai},2}^n$  defined as

$$(1.4) \quad \rho_{\text{Ai},2}^n(\mathbf{x}_n) = \det[K_{\text{Ai},2}(x_i, x_j)]_{i,j=1}^n.$$

Here  $K_{\text{Ai},2}$  is the continuous kernel given by, for  $x \neq y$ ,

$$(1.5) \quad K_{\text{Ai},2}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y},$$

where we set  $\text{Ai}'(x) = d\text{Ai}(x)/dx$  and denote by  $\text{Ai}(\cdot)$  the Airy function such that

$$(1.6) \quad \text{Ai}(z) = \frac{1}{2\pi} \int_{\mathbb{R}} dk e^{i(zk + k^3/3)}, \quad z \in \mathbb{R}.$$

For  $\beta = 1, 4$  similar expressions in terms of the quaternion determinant are obtained as shown in (2.1).

From (1.2) we obtain the associated stochastic dynamics  $\mathbf{X}^n = (X_t^{n,i})_{i=1}^n$  from the stochastic differential equation (SDE):

$$dX_t^{n,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^n \frac{1}{X_t^{n,i} - X_t^{n,j}} dt - \frac{\beta}{4n} X_t^{n,i} dt \quad (i = 1, \dots, n).$$

For  $\beta = 1, 2$  and  $4$ , this SDE was introduced by Dyson and is referred to as the equation for Dyson's Brownian motions. Taking  $n \rightarrow \infty$ , we can naturally obtain the infinite-dimensional SDE (ISDE)

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^{\infty} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{Z}).$$

This ISDE (with  $\beta = 2$ ) is often called Dyson's model (in infinite dimensions), and was introduced by Spohn at the heuristic level. Spohn [39] constructed the associated unlabeled dynamics as an  $L^2$  Markovian semi-group by introducing a Dirichlet form related to  $\mu_{\text{sin},2}$ .

In [28, 29], not only for  $\beta = 2$ , but also for  $\beta = 1, 4$ , the first author constructed the  $\mu_{\text{sin},\beta}$ -reversible unlabeled diffusion  $\mathbf{X}_t = \sum_{i \in \mathbb{Z}} \delta_{X_t^i}$ . He proved that tagged particles  $X_t^i$  never collide with one another and that the associated labeled system  $\mathbf{X}_t = (X_t^i)_{i \in \mathbb{Z}}$  solves the ISDE

$$(1.7) \quad dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \sum_{\substack{j \neq i \\ |X_t^i - X_t^j| < r}} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{Z}).$$

We therefore have infinitely many, noncolliding paths describing the motions of a limit particle system as a  $\mathbb{R}^{\mathbb{Z}}$ -valued diffusion process. We remark that since  $\mu_{\text{sin},\beta}$  is translation invariant, only conditional convergence is possible for the sum in (1.7). Because of the conditional convergence, the shape of the limit SDEs is quite sensitive.

In soft-edge scaling, we obtain from (1.3) the  $n$ -particle dynamics  $\mathbf{X}^n = (X_t^{n,i})_{i=1}^n$  given by the SDE:

$$(1.8) \quad dX_t^{n,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^n \frac{1}{X_t^{n,i} - X_t^{n,j}} dt - \frac{\beta}{2} \left\{ n^{1/3} + \frac{1}{2n^{1/3}} X_t^{n,i} \right\} dt.$$

Because of the divergence of the second and third terms on the right-hand side as  $n \rightarrow \infty$ , no simple guess of the limit SDE is possible.

The purpose of this paper is to detect and solve the limit ISDE at the soft-edge scaling. In fact, from (1.8), we derive the ISDE

$$(1.9) \quad dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \left( \sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|x| < r} \frac{\hat{\varrho}(x)}{-x} dx \right\} dt.$$

Here we set

$$(1.10) \quad \hat{\varrho}(x) = \frac{1_{(-\infty, 0)}(x)}{\pi} \sqrt{-x}.$$

We prove that ISDE (1.9) has a strong solution  $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$  for  $\mu_{\text{Ai}, \beta}$ -a.s.  $\mathbf{s} = \sum_{i=1}^{\infty} \delta_{X_0^i}$  and strong uniqueness of the solutions in one of the main theorems (Theorem 2.3). We solve ISDE (1.9) through an equivalent yet more refined representation of the ISDE

$$\begin{aligned} dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \left( \sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|y| < r} \frac{\rho_{\text{Ai}, \beta, X_t^i}^1(y)}{X_t^i - y} dy \right\} dt \\ + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \int_{|y| < r} \frac{\rho_{\text{Ai}, \beta, X_t^i}^1(y)}{X_t^i - y} dy - \int_{|y| < r} \frac{\hat{\varrho}(y)}{-y} dy \right\} dt. \end{aligned}$$

Here  $\rho_{\text{Ai}, \beta, x}^1$  is a one-correlation function of the reduced Palm measure  $\mu_{\text{Ai}, \beta, x}$  conditioned at  $x$ . Note that the second term on the right-hand side is neutral, and the coefficient of the third term can be regarded as the  $-\frac{1}{2}$  multiple of the derivative of the free potential  $\Phi_\beta$  in Theorem 5.6.

An outline of the derivation of (1.9) is given below. Taking into account the inverse of the soft-edge scaling

$$x \mapsto x n^{-1/6} + 2\sqrt{n} = \sqrt{n}(x n^{-2/3} + 2),$$

we set

$$(1.11) \quad \hat{\varrho}^n(x) = n^{1/3} \sigma_{\text{semi}}(x n^{-2/3} + 2).$$

From (1.11) we take  $\hat{\varrho}^n$  as the first approximation of the one-correlation function  $\rho_{\text{Ai}, \beta, x}^{n,1}$  of the reduced Palm measure  $\mu_{\text{Ai}, \beta, x}^n$  of the  $n$ -particle Airy RPF  $\mu_{\text{Ai}, \beta}^n$ .

From (1.11) we see that

$$(1.12) \quad \int_{\mathbb{R}} \hat{\varrho}^n(x) dx = n.$$

The prefactor  $n^{1/3}$  in the definition of  $\hat{\varrho}^n(x) = n^{1/3}\sigma(xn^{-2/3} + 2)$  is chosen for (1.12). A simple calculation shows that

$$(1.13) \quad \hat{\varrho}^n(x) = \frac{1_{(-4n^{2/3}, 0)}(x)}{\pi} \sqrt{-x \left(1 + \frac{x}{4n^{2/3}}\right)},$$

$$(1.14) \quad \lim_{n \rightarrow \infty} \hat{\varrho}^n(x) = \hat{\varrho}(x) \quad \text{compact uniformly.}$$

The key point of the derivation is that

$$(1.15) \quad n^{1/3} = \int_{\mathbb{R}} \frac{\hat{\varrho}^n(x)}{-x} dx.$$

Equations (1.13) and (1.14) then justify the appearance of  $\hat{\varrho}(x)$  in the limit ISDE (1.9). Indeed, we prove that, as  $n \rightarrow \infty$ ,

$$\begin{aligned} dX_t^i &\sim dB_t^i + \frac{\beta}{2} \left\{ \left( \sum_{j \neq i, j=1}^n \frac{1}{X_t^i - X_t^j} \right) - n^{1/3} \right\} dt \\ &\sim dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \left( \sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|x| < r} \frac{\hat{\varrho}^n(x)}{-x} dx \right\} dt \quad \text{by (1.15)} \\ &\sim dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \left( \sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|x| < r} \frac{\hat{\varrho}(x)}{-x} dx \right\} dt \quad \text{by (1.14),} \end{aligned}$$

thereby obtaining the ISDE (1.9). We also note that if we replace  $\hat{\varrho}$  by  $1/\pi$ , then we obtain the ISDE (1.7).

Systems described by ISDEs of the form

$$dX_t^i = dB_t^i - \frac{1}{2} \nabla \Phi(X_t^i) - \frac{1}{2} \sum_{j=1, j \neq i}^{\infty} \nabla \Psi(X_t^i, X_t^j) dt \quad (i \in \mathbb{Z}).$$

are called interacting Brownian motions in infinite dimensions. They describe infinitely many Brownian particles moving in  $\mathbb{R}^d$  with free potential  $\Phi = \Phi(x)$  and interaction potential  $\Psi = \Psi(x, y)$ . The study of interacting Brownian motions in infinite dimensions is initiated by Lang [18, 19], and continued by Fritz [8], the second author [40], and others. In their works, interaction potentials  $\Psi(x, y) = \Psi(x - y)$  are of class  $C_0^3$  or exponentially decay at infinity. Such a restriction on  $\Psi$  excludes logarithmic potentials.

We use a general theory presented in [28] to solve (1.9) in Theorem 2.2. The solution at this stage has the usual meaning; that is, a pair of infinite-dimensional processes  $(\mathbf{X}, \mathbf{B})$  satisfying (1.9). We also prove that no particles of the solution  $\mathbf{X} = (X_t^i)_{i \in \mathbb{N}}$  collide with one another for all  $t$ . That is,

$$P(X_t^i \neq X_t^j \text{ for all } 0 \leq t < \infty, i \neq j) = 1.$$

Hence, we always label the particles in such a way that  $X_t^i > X_t^j$  for all  $i < j$ . Therefore, the right-most particle is denoted by  $X_t^1$ . The solution  $(X_t^i)_{i \in \mathbb{N}}$  is a  $\mathbb{R}_{>}^{\mathbb{N}}$ -valued process, where  $\mathbb{R}_{>}^{\mathbb{N}} = \{(x_i) \in \mathbb{R}^{\mathbb{N}}; x_i > x_j \text{ } (i < j)\}$ .

In the co-paper [32], we have developed a general theory on the existence and uniqueness of strong solutions for ISDEs of a general class of interactions including logarithmic potentials. Applying the theory, we refine the results much further in Theorem 2.3, which states the existence and the pathwise uniqueness of strong solutions of (1.9). Here as usual a strong solution means that  $\mathbf{X}$  is a function of the (given) Brownian motion  $\mathbf{B}$ , with the uniqueness explained in Theorem 2.3.

The uniqueness of the solutions of the ISDE yields several significant consequences, such as the uniqueness of quasi-regular Dirichlet forms and (suitably formulated) martingale problems. The most important one is the identity between our construction and the algebraic construction based on space-time correlation functions where  $\beta = 2$  (Theorem 2.4). When  $\beta = 2$ , natural infinite-dimensional stochastic dynamics have been constructed using the extended Airy kernel [7, 21, 14], which we refer as the algebraic construction in the above. Such a construction has been studied by Prähofer-Spohn [36] and Johansson [13], amongst others. In these works, the construction of the dynamics of the top particle  $\mathcal{A}(t)$ , which was called the Airy process in Prähofer-Spohn [36], has attracted much attention. In [34], we prove that these stochastic dynamics also satisfy ISDE (1.9). From the uniqueness of the solutions of (1.9), we deduce that these two stochastic dynamics are the same. Hence, in particular, the right-most particle  $X_t^1$  is treated with the Airy process  $\mathcal{A}(t)$ .

When  $\beta = 1, 4$ , no algebraic construction of the stochastic dynamics is known. Our construction based on stochastic analysis is valid even for the cases where  $\beta = 1, 4$ .

We further prove in Theorem 2.6 the Cameron-Martin formula for the solution of SDE (1.9). As an application, we obtain each tagged particle in a semi-martingale manner, and the local properties of their trajectories are similar to those of Brownian motions. We have then an immediate affirmative answer for Johansson's conjecture [13, Conjecture 1.5] that  $H(t) = \mathcal{A}(t) - t^2$  almost surely has a unique maximum point in  $[-T, T]$ . This conjecture was discussed and solved by Corwin [4] and Hägg [10] using a different method.

The paper is organized as follows. In Section 2, we define the problems and state our main theorems (Theorem 2.1–Theorem 2.3, and Theorem 2.6). In Section 3 we prepare a general theory from [32] on ISDEs describing interacting Brownian motions. Various estimates of finite-particle systems approximating Airy RPFs are investigated in Section 4. In Subsection 5.1–Subsection 5.4 we prove the main theorems Theorem 2.1–Theorem 2.3, and Theorem 2.6, respectively. In Subsection 6.1 (Appendix 1), we recall the definition of a quaternion determinant. In Subsection 6.2 (Appendix 2), Subsection 6.3 (Appendix 3), and Subsection 6.4 (Appendix 4), we

give some estimates of Airy kernels, kernels of  $n$ -particles, and Hermite polynomials, respectively. In Subsection 6.5 (Appendix 5), we complete the proof of Lemma 4.5.

## 2 Main results

This section defines the problem and states the main theorems.

We begin by recalling the notion of the configuration space over  $\mathbb{R}$ . Let

$$\mathbf{S} = \{\mathbf{s} = \sum_i \delta_{s_i} ; \mathbf{s}(K) < \infty \text{ for all compact sets } K \subset \mathbb{R}\},$$

where  $\delta_a$  denotes the delta measure at  $a$ . We endow  $\mathbf{S}$  with the vague topology, under which  $\mathbf{S}$  is a Polish space.  $\mathbf{S}$  is called the configuration space over  $\mathbb{R}$ .

A probability measure  $\mu$  on  $(\mathbf{S}, \mathcal{B}(\mathbf{S}))$  is called the RPF on  $\mathbb{R}$ . To define Airy RPFs, we recall the notion of correlation functions.

A symmetric locally integrable function  $\rho^n : \mathbb{R}^n \rightarrow [0, \infty)$  is called the  $n$ -point correlation function of an RPF  $\mu$  on  $\mathbf{S}$  w.r.t. the Lebesgue measure if  $\rho^n$  satisfies

$$\int_{A_1^{k_1} \times \dots \times A_m^{k_m}} \rho^n(x_1, \dots, x_n) dx_1 \cdots dx_n = \int_{\mathbf{S}} \prod_{i=1}^m \frac{\mathbf{s}(A_i)!}{(\mathbf{s}(A_i) - k_i)!} d\mu$$

for any sequence of disjoint bounded measurable subsets  $A_1, \dots, A_m \subset \mathbb{R}$  and a sequence of natural numbers  $k_1, \dots, k_m$  satisfying  $k_1 + \dots + k_m = n$ . When  $\mathbf{s}(A_i) - k_i < 0$ , according to our interpretation,  $\mathbf{s}(A_i)! / (\mathbf{s}(A_i) - k_i)! = 0$  by convention. It is known that  $\{\rho^n\}_{n \in \mathbb{N}}$  determines the measure  $\mu$  under a weak condition. In particular, determinantal RPFs generated by given kernels and reference measures are uniquely given [38].

We denote by  $\mu_{\text{Ai},2}$  the RPF whose correlation functions are given by (1.4), and call it the Airy<sub>2</sub> RPF. RPFs  $\mu_{\text{Ai},1}$  and  $\mu_{\text{Ai},4}$  are defined similarly using quaternions. By definition,  $\mu_{\text{Ai},\beta}$  ( $\beta = 1, 4$ ) are RPFs with  $n$ -correlation functions  $\rho_{\text{Ai},\beta}^n$  given by

$$(2.1) \quad \rho_{\text{Ai},\beta}^n(x_1, \dots, x_n) = \text{qdet}[K_{\text{Ai},\beta}(x_i, x_j)]_{i,j=1}^n.$$

Here  $\text{qdet}$  denotes the quaternion determinant defined by (6.1), and the  $K_{\text{Ai},\beta}$  are quaternion-valued kernels defined by (6.3).

For a subset  $A \subset \mathbb{R}$ , we define the map  $\pi_A : \mathbf{S} \rightarrow \mathbf{S}$  by  $\pi_A(\mathbf{s}) = \mathbf{s}(\cdot \cap A)$ . We say a function  $f : \mathbf{S} \rightarrow \mathbb{R}$  is local if  $f$  is  $\sigma[\pi_A]$ -measurable for some compact set  $A$ .

Let  $\mathbf{u}$  be the map defined on  $\{\sum_{k=1}^{\infty} \mathbb{R}^k\} \cup \mathbb{R}^{\mathbb{N}}$  such that  $\mathbf{u}((x_i)) = \sum_i \delta_{x_i}$ . For a local function  $f : \mathbf{S} \rightarrow \mathbb{R}$ , there exists a unique symmetric function  $\tilde{f}$  on  $\mathbf{u}^{-1}(\mathbf{S})$  such that  $f(\mathbf{s}) = \tilde{f}((s_i))$  for  $\mathbf{s} = \sum_i \delta_{s_i}$ . We say a local function  $f : \mathbf{S} \rightarrow \mathbb{R}$  is smooth if  $\tilde{f}$  is smooth.

We introduce the natural square field on  $\mathbf{S}$  and Dirichlet forms for given RPF  $\mu$ . Let  $\mathcal{D}_\circ$  be the set of all local, smooth functions on  $\mathbf{S}$ . For  $f, g \in \mathcal{D}_\circ$ , we set  $\mathbb{D}[f, g] : \mathbf{S} \rightarrow \mathbb{R}$  according to

$$\mathbb{D}[f, g](\mathbf{s}) = \frac{1}{2} \sum_i \frac{\partial_i \tilde{f}(\mathbf{s})}{\partial s_i} \frac{\partial_i \tilde{g}(\mathbf{s})}{\partial s_i},$$

where  $\mathbf{s} = \sum_i \delta_{s_i}$  and  $\mathbf{s} = (s_i)$ . Let  $\mathcal{E}^\mu$  be the bilinear form defined as

$$\mathcal{E}^\mu(f, g) = \int_{\mathbf{S}} \mathbb{D}[f, g] d\mu$$

with domain  $\mathcal{D}_\circ^\mu = \{f \in \mathcal{D}_\circ; \mathcal{E}^\mu(f, f) < \infty, f \in L^2(\mathbf{S}, \mu)\}$ .

Let  $\Lambda$  denote the Poisson RPF whose intensity is the Lebesgue measure. If  $\mu = \Lambda$ , then the bilinear form  $(\mathcal{E}^\Lambda, \mathcal{D}_\circ^\Lambda)$  is closable on  $L(\mathbf{S}, \Lambda)$ , and the closure is of a quasi-regular Dirichlet form. The associated diffusion  $\mathbb{B}_t^\mathbf{s} = \sum_{i \in \mathbb{N}} \delta_{B_t^i + s_i}$  is the  $\mathbf{S}$ -valued Brownian motion starting at  $\mathbf{s} = \sum_i \delta_{s_i}$ . In fact,  $\{B^i\}_{i \in \mathbb{N}}$ , where  $B^i = \{B_t^i\}_{[0, \infty)}$ , are independent copies of one-dimensional standard Brownian motion [25]. It is thus natural to ask, if we replace  $\Lambda$  by  $\mu_{\text{Ai}, \beta}$  ( $\beta = 1, 2, 4$ ), whether the forms  $(\mathcal{E}^{\mu_{\text{Ai}, \beta}}, \mathcal{D}_\circ^{\mu_{\text{Ai}, \beta}})$  are still closable on  $L^2(\mathbf{S}, \mu_{\text{Ai}, \beta})$  and associated diffusions exist.

We write  $\mathbf{s}(x) = \mathbf{s}(\{x\})$ . Let

$$(2.2) \quad \begin{aligned} \mathbf{S}_{\text{s.i.}} &= \{\mathbf{s} \in \mathbf{S}; \mathbf{s}(x) \leq 1 \text{ for all } x \in \mathbb{R}, \mathbf{s}(\mathbb{R}) = \infty\}, \\ \mathbf{S}_{\text{s.i.}}^{+f} &= \{\mathbf{s} \in \mathbf{S}_{\text{s.i.}}; \mathbf{s}(\mathbb{R}^+) < \infty\}. \end{aligned}$$

By definition,  $\mathbf{S}_{\text{s.i.}}$  is the set of configurations consisting of an infinite number of single-point measures, and  $\mathbf{S}_{\text{s.i.}}^{+f}$  is its subset with only a finite number in  $\mathbb{R}^+$ . It is well known that

$$(2.3) \quad \mu_{\text{Ai}, \beta}(\mathbf{S}_{\text{s.i.}}^{+f}) = 1 \quad \text{for } \beta = 1, 2, 4.$$

From (2.2) and (2.3), for  $\mu_{\text{Ai}, \beta}$ -a.s.  $\mathbf{s} = \sum_i \delta_{s_i}$ , we can and do label  $\{s_i\}$  in such a way that  $s_i > s_j$  for all  $i < j$ . Therefore, let  $\mathbb{R}_{>}^\mathbb{N} = \{(x_i)_{i \in \mathbb{N}}; x_i > x_j \text{ for all } i < j\}$  and define the map  $\mathfrak{l} : \mathbf{S}_{\text{s.i.}}^{+f} \rightarrow \mathbb{R}_{>}^\mathbb{N}$  by

$$(2.4) \quad \mathfrak{l}(\mathbf{s}) = (s_1, s_2, \dots), \text{ where } \mathbf{s} = \sum_{i=1}^{\infty} \delta_{s_i}.$$

We see that  $\mu_{\text{Ai}, \beta}$  can be regarded as probability measures on  $\mathbb{R}_{>}^\mathbb{N}$  according to  $\mu_{\text{Ai}, \beta} \circ \mathfrak{l}^{-1}$ .

We call  $\mathfrak{l}$  a label. In general, there exist infinitely many different types of labels. However, we always take the label as above because this choice is clearly the most



natural for the Airy RPF. We remark that for other RPFs such as sine and Ginibre RPFs, there is generally no such canonical choice of labels.

Let  $\mathbf{A} \subset \mathbf{S}_{\text{s.i.}}$ . Let  $C([0, \infty); \mathbf{A})$  be the set of  $\mathbf{A}$ -valued continuous paths. The element  $\mathbf{X} \in C([0, \infty); \mathbf{A})$  can be written as  $\mathbf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$ , where  $X^i \in C(I^i; \mathbb{R})$  with (possibly random) interval  $I^i$  of the form  $I^i = [0, b^i)$  or  $I^i = (a^i, b^i)$  ( $0 \leq a^i < b^i \leq \infty$ ). We take each interval  $I$  to be the maximal one. The expression  $\mathbf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$  is then unique up to labeling because of the assumption  $\mathbf{A} \subset \mathbf{S}_{\text{s.i.}}$ . Let  $C_{\text{ne}}([0, \infty); \mathbf{A})$  be the subset of  $C([0, \infty); \mathbf{A})$  consisting of non-explosive paths. Then  $I^i = [0, \infty)$  for all  $i \in \mathbb{N}$  and

$$C_{\text{ne}}([0, \infty); \mathbf{A}) = \{\mathbf{X} \in C([0, \infty); \mathbf{A}); \mathbf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}, X^i \in C([0, \infty), \mathbb{R}) \ (\forall i)\}.$$

We remark that generally  $C_{\text{ne}}([0, \infty); \mathbf{A}) \neq C([0, \infty); \mathbf{A})$  because we equip  $\mathbf{S}$  with a vague topology. The advantage of considering the set of non-explosive and noncolliding paths is that we can naturally relate the labeled path  $\mathbf{X} \in C([0, \infty); \mathbb{R}^{\mathbb{N}})$  to each  $\mathbf{X} \in C_{\text{ne}}([0, \infty); \mathbf{S}_{\text{s.i.}})$  as follows.

Let  $\mathfrak{l}_{\text{path}}$  be the map from  $C_{\text{ne}}([0, \infty); \mathbf{S}_{\text{s.i.}}^{+f})$  to  $C([0, \infty); \mathbb{R}_{>}^{\mathbb{N}})$  defined as

$$(2.5) \quad \mathfrak{l}_{\text{path}}(\mathbf{X}) = \{(X_t^i)_{i \in \mathbb{N}}\}_{t \in [0, \infty)}, \text{ where } \mathbf{X} = \left\{ \sum_{i=1}^{\infty} \delta_{X_t^i} \right\}_{t \in [0, \infty)}.$$

We write  $\mathbf{X} = \mathfrak{l}_{\text{path}}(\mathbf{X})$ , and call  $\mathbf{X}$  (resp.  $\mathbf{X}$ ) a labeled (resp. unlabeled) process.

To state the main theorems, we recall the terminology for diffusion processes in a general framework. For a Polish space  $S$ , we say a family of probability measures  $\{P_s\}_{s \in S}$  on  $C([0, \infty); S)$  is a conservative diffusion with state space  $S$  if, under  $P_s$ , the canonical process  $\{(X_t, P_s)\}$  has a strong Markov property and  $X_0 = s$ . In general, a diffusion may explode, and is defined until life time  $\tau$ . Since we consider a conservative diffusion,  $\tau = \infty$ . By construction,  $\{X_t\}$  is a continuous process. We say a diffusion is  $\mu$ -reversible if it has an invariant probability measure  $\mu$  and is symmetric with respect to  $\mu$ . For a given closable nonnegative form  $(\mathcal{E}, \mathcal{D}_0)$  on  $L^2(S, \mu)$ , we say a conservative diffusion  $\{P_s\}_{s \in S_0}$  with state space  $S_0$  is associated with  $(\mathcal{E}, \mathcal{D}, L^2(S, \mu))$  if  $\mu(S_0^c) = 0$  and  $E_s[f(X_t)] = T_t f(s)$   $\mu$ -a.e  $s \in S_0$  for any  $f \in L^2(S, \mu)$  and for all  $t$ . Here  $(\mathcal{E}, \mathcal{D})$  is the closure of  $(\mathcal{E}, \mathcal{D}_0)$  on  $L^2(S, \mu)$ , and  $T_t$  is the associated  $L^2$ -semi group. Since the diffusion  $\{P_s\}_{s \in S_0}$  is conservative, we see that  $P_s(X_t \in S_0 \text{ for all } t) = 1$  for all  $s \in S_0$ . Such a closed form automatically becomes a local Dirichlet form and, by construction,  $T_t$  is a Markovian semi-group (see [9]).

We state our first main theorem.

**Theorem 2.1.** *Assume  $\beta = 1, 2, 4$ . Then:*

(1) *The bilinear form  $(\mathcal{E}^{\mu_{\text{Ai}, \beta}}, \mathcal{D}_{\circ}^{\mu_{\text{Ai}, \beta}})$  is closable on  $L^2(\mathbf{S}, \mu_{\text{Ai}, \beta})$ .*

(2) There exists a diffusion  $\{\mathbf{P}_s\}_{s \in S}$  associated with  $(\mathcal{E}^{\mu_{\text{Ai},\beta}}, \mathcal{D}^{\mu_{\text{Ai},\beta}})$ , which is the closure of  $(\mathcal{E}^{\mu_{\text{Ai},\beta}}, \mathcal{D}_o^{\mu_{\text{Ai},\beta}})$  on  $L^2(S, \mu_{\text{Ai},\beta})$ .

(3) There exists a measurable subset  $S_{\mu_{\text{Ai},\beta}} \subset S$  satisfying the following:

$$(2.6) \quad S_{\mu_{\text{Ai},\beta}} \subset S_{\text{s.i.}}^+, \quad \mu_{\text{Ai},\beta}(S_{\mu_{\text{Ai},\beta}}) = 1,$$

$$(2.7) \quad \mathbf{P}_s(X_t \in S_{\mu_{\text{Ai},\beta}} \text{ for all } 0 \leq t < \infty) = 1 \quad \text{for all } s \in S_{\mu_{\text{Ai},\beta}},$$

$$(2.8) \quad \mathbf{P}_s(\sup_{t \in [0, T]} |X_t^i| < \infty \text{ for all } T, i \in \mathbb{N}) = 1 \quad \text{for all } s \in S_{\mu_{\text{Ai},\beta}}.$$

Here  $\mathbf{X} = (X^i)_{i \in \mathbb{N}} = \mathbf{l}_{\text{path}}(\mathbf{X})$  is the labeled process given by (2.5).

**Remark 2.1.** From (2.7) we deduce that  $\{\mathbf{P}_s\}_{s \in S_{\mu_{\text{Ai},\beta}}}$  is a diffusion with state space  $S_{\mu_{\text{Ai},\beta}}$ . Moreover, from (2.6) we see that  $\{\mathbf{P}_s\}_{s \in S_{\mu_{\text{Ai},\beta}}}$  is reversible with invariant probability measure  $\mu_{\text{Ai},\beta}(\cdot \cap S_{\mu_{\text{Ai},\beta}})$ .

Next, we solve the ISDE (1.9). Let  $\mathbf{l}$  and  $S_{\mu_{\text{Ai},\beta}}$  be as in (2.4) and Theorem 2.1, respectively. Let  $S_{\mu_{\text{Ai},\beta}} = \mathbf{l}(S_{\mu_{\text{Ai},\beta}})$  and  $\mathbf{P}_s = \mathbf{P}_s \circ \mathbf{l}_{\text{path}}^{-1}$ , where  $s = \mathbf{l}(s)$ . From Theorem 2.1 (3) we deduce that

$$\mathbf{P}_s(C([0, \infty); \mathbb{R}_{>}^{\mathbb{N}})) = 1.$$

Hence,  $\mathbf{X} = \mathbf{l}_{\text{path}}(\mathbf{X}) \in C([0, \infty); \mathbb{R}_{>}^{\mathbb{N}})$ . We call  $\mathbf{B} = (B^i)_{i \in \mathbb{N}}$  an  $\mathbb{R}^{\mathbb{N}}$ -valued Brownian motion if the  $\{B^i\}_{i \in \mathbb{N}}$  are independent copies of the standard Brownian motion on  $\mathbb{R}$ .

**Theorem 2.2.** Assume  $\beta = 1, 2, 4$ . Let  $S_{\mu_{\text{Ai},\beta}}$  and  $\mathbf{P}_s$  be as above. Then:

(1) Let  $s \in S_{\mu_{\text{Ai},\beta}}$ . Under  $\mathbf{P}_s$ , the canonical process  $\mathbf{X} = \{(X_t^i)_{i \in \mathbb{N}}\}$  is a solution of ISDE (1.9) starting at  $s = (s_i)_{i \in \mathbb{N}}$ . That is, there exists an  $\mathbb{R}^{\mathbb{N}}$ -valued Brownian motion  $\mathbf{B}$  defined on  $(C([0, \infty); \mathbb{R}^{\mathbb{N}}), \mathbf{P}_s)$  such that the pair  $(\mathbf{X}, \mathbf{B})$  satisfies

$$(1.9) \quad dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \sum_{\substack{|X_t^j| < r \\ j \neq i}} \frac{1}{X_t^i - X_t^j} - \int_{|x| < r} \frac{\hat{\rho}(x)}{-x} dx \right\} dt \quad (i \in \mathbb{N})$$

$$\mathbf{X}_0 = s.$$

(2)  $\{\mathbf{P}_s\}_{s \in S_{\mu_{\text{Ai},\beta}}}$  is a  $\mu_{\text{Ai},\beta} \circ \mathbf{l}^{-1}$ -reversible diffusion with state space  $S_{\mu_{\text{Ai},\beta}}$ .

(3) The distribution of  $X_t^1$  under  $\mathbf{P}_{\mu_{\text{Ai},\beta} \circ \mathbf{l}^{-1}}$  is a  $\beta$  Tracy-Widom distribution ([1, pp 92-94]).

**Remark 2.2.** The reversibility of the labeled dynamics Theorem 2.2 (2) is specific in Airy interacting Brownian motions, since there exist simple bijections between the support of the RPFs in configuration spaces and the labeled space  $\mathbb{R}_{>}^{\mathbb{N}}$ . In the case of Bessel interacting Brownian motions [11], there also exists a canonical bijection between the support of the Bessel RPFs and the labeled space  $[0, \infty)_{<}^{\mathbb{N}} = \{(x_i)_{i=1}^{\infty} \in [0, \infty)^{\mathbb{N}}; 0 \leq x_i < x_j \text{ for all } i < j\}$ .

In [32], we developed a general theory of the uniqueness and existence of a strong solution of ISDEs concerning interacting Brownian motions including (1.9), Dyson's model, and Ginibre interacting Brownian motions, amongst others. Using this theory together with the results obtained in this paper, we refine the meaning of ISDE (1.9) in the next theorem.

Let  $\mathcal{T}(\mathbf{S})$  be the tail  $\sigma$ -field of  $\mathbf{S}$  defined by (3.15). We say that a RPF  $\nu$  is tail trivial if  $\nu(\mathbf{A}) \in \{0, 1\}$  for all  $\mathbf{A} \in \mathcal{T}(\mathbf{S})$ . Let  $\mu_{\mathbf{A}_i, \beta, \mathbf{t}}(\cdot) = \mu_{\mathbf{A}_i, \beta}(\cdot | \mathcal{T}(\mathbf{S}))(\mathbf{t})$  be a regular conditional probability. It is known that  $\mu_{\mathbf{A}_i, \beta, \mathbf{t}}$  is tail trivial for  $\mu_{\mathbf{A}_i, \beta}$ -a.s.  $\mathbf{t} \in \mathbf{S}$  (see Lemma 3.4).

We say a family of solutions  $(\mathbf{X}, \mathbf{B})$  satisfies the  $\nu$ -absolute continuity condition if the associated unlabeled process  $\mathbf{X} = \{\mathbf{X}_t\}$  satisfies

$$(2.9) \quad \mathbf{P}_\nu \circ \mathbf{X}_t \prec \nu \quad \text{for all } t.$$

Here  $\mathbf{P}_\nu = \int_{\mathbf{S}} \mathbf{P}_s \nu(ds)$ ,  $\mathbf{P}_s = \mathbf{P}_{\mathbf{u}(s)}$ , and  $\mathbf{P}_s$  is the distribution of  $\mathbf{X}$  starting at  $\mathbf{s}$ .

For an  $\mathbb{R}^N$ -valued Brownian motion  $\mathbf{B}$  starting at the origin, we denote by  $P^{\mathbf{B}}$  its distribution.

**Theorem 2.3.** *Assume  $\beta = 1, 2, 4$ . Then the following holds.*

- (1) **Existence of strong solutions:** *ISDE (1.9) has strong solutions  $\mathbf{X}(\cdot, \mathbf{s})$  starting at  $\mu_{\mathbf{A}_i, \beta} \circ \mathfrak{l}^{-1}$ -a.e.  $\mathbf{s}$ , satisfying the  $\mu_{\mathbf{A}_i, \beta, \mathbf{t}}$ -absolute continuity condition (2.9) and being  $\mu_{\mathbf{A}_i, \beta, \mathbf{t}} \circ \mathfrak{l}^{-1}$ -reversible diffusions for  $\mu_{\mathbf{A}_i, \beta}$ -a.s.  $\mathbf{t}$ .*
- (2) **Strong uniqueness:** *Suppose that  $(\mathbf{X}, \mathbf{B})$  and  $(\hat{\mathbf{X}}, \mathbf{B})$  are solutions of (1.9) with the same Brownian motion  $\mathbf{B}$  satisfying the  $\nu$ -absolute continuity condition (2.9). If  $\nu$  is tail trivial, then  $\mathbf{X}$  and  $\hat{\mathbf{X}}$  are strong solutions and*

$$P^{\mathbf{B}}(\mathbf{X}(\cdot, \mathbf{s}) = \hat{\mathbf{X}}(\cdot, \mathbf{s})) = 1 \quad \text{for } \nu \circ \mathfrak{l}^{-1}\text{-a.e. } \mathbf{s}.$$

*In particular, there exists a unique strong solution satisfying the  $\mu_{\mathbf{A}_i, \beta, \mathbf{t}}$ -absolute continuity condition for  $\mu_{\mathbf{A}_i, \beta}$ -a.s.  $\mathbf{t}$ .*

**Remark 2.3.** (1) *Strong solutions in Theorem 2.3 (1) mean that, for a given Brownian motion  $\mathbf{B}$  and initial point  $\mathbf{s} \in \mathbf{S}_{\mu_{\mathbf{A}_i, \beta}}$ , there exists a function  $\mathbf{X} = \mathbf{X}(\mathbf{B}, \mathbf{s})$  of  $(\mathbf{B}, \mathbf{s})$  such that the pair  $(\mathbf{X}, \mathbf{B})$  satisfies ISDE (1.9).*

(2) *Theorem 2.3 (2) implies that if  $(\mathbf{X}, \mathbf{B})$  is a solution of (1.9) starting at  $\nu \circ \mathfrak{l}^{-1}$ -a.s.  $\mathbf{s}$  satisfying the  $\nu$ -absolute continuity condition (2.9), and if  $\nu$  is tail trivial, then  $(\mathbf{X}, \mathbf{B})$  is a strong solution for  $\nu \circ \mathfrak{l}^{-1}$ -a.s. starting point  $\mathbf{s}$ . In this sense, ISDE (1.9) has strong uniqueness.*

(3) *We have proved the unique existence of strong solutions satisfying  $\mu_{\mathbf{A}_i, \beta, \mathbf{t}}$ -absolute continuity condition for  $\mu_{\mathbf{A}_i, \beta}$ -a.s.  $\mathbf{t}$ . This result does not exclude the possibility that a solution not satisfying the absolute continuity condition exists. Such solutions, if they exist, would change the tail.*

If  $\beta = 2$ , the infinite-dimensional stochastic dynamics can be constructed using the space-time correlation functions [7, 36, 13, 21, 14]. Let  $\mathbb{K}_{\text{Ai}}(s, x, t, y)$  be the extended Airy kernel defined as

$$\mathbb{K}_{\text{Ai}}(s, x, t, y) = \begin{cases} \int_{-\infty}^0 du e^{(t-s)u/2} \text{Ai}(x-u) \text{Ai}(y-u), & \text{if } s \leq t \\ - \int_0^{\infty} du e^{(t-s)u/2} \text{Ai}(x-u) \text{Ai}(y-u), & \text{if } s > t. \end{cases}$$

The determinantal process  $\mathbf{Y} = \{\mathbf{Y}_t\}$  with the extended kernel  $\mathbb{K}_{\text{Ai}}(s, x, t, y)$  is an  $\mathbf{S}$ -valued process such that, for any  $M \in \mathbb{N}$ ,  $\mathbf{f} = (f_1, f_2, \dots, f_M) \in C_0(\mathbb{R})^M$  and a sequence of times  $\mathbf{t} = (t_1, t_2, \dots, t_M)$  with  $0 < t_1 < \dots < t_M < \infty$ , if we set  $\chi_{t_m}(x) = e^{f_m(x)} - 1$ ,  $1 \leq m \leq M$ , the moment generating function of a multi-time distribution,

$$\Psi^{\mathbf{t}}[\mathbf{f}] \equiv \mathbb{E} \left[ \exp \left\{ \sum_{m=1}^M \int_{\mathbb{R}} f_m(x) \mathbf{Y}_{t_m}(dx) \right\} \right],$$

is given by a Fredholm determinant

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \underset{\substack{(s,t) \in \{t_1, t_2, \dots, t_M\}^2, \\ (x,y) \in \mathbb{R}^2}}{\text{Det}} \left[ \delta_{st} \delta(x-y) + \mathbb{K}_{\text{Ai}}(s, x, t, y) \chi_t(y) \right].$$

The reader may refer to [15] for the details.

Finite- and infinite-dimensional determinantal processes were introduced as multi-matrix models [5, 22], tiling models [12], and surface growth models [36], and have been studied extensively in [3, 2], among others. We remark that the Markov property of infinite-dimensional determinantal processes as above is highly nontrivial, unlike that of the infinite-dimensional stochastic dynamics given by the Dirichlet form approach.

Let  $\mathbf{Q}_{\mathbf{s}}$  be the distribution of a determinantal process with the extended Airy kernel starting at  $\mathbf{s}$  [16]. It is known [17] that there exists  $\mathbf{S}_0$  such that  $\mu_{\text{Ai},2}(\mathbf{S}_0) = 1$  and that  $\{\mathbf{Q}_{\mathbf{s}}\}_{\mathbf{s} \in \mathbf{S}_0}$  is a continuous, stationary Markov process. In [34], we refine this result in such a way that  $\{\mathbf{Q}_{\mathbf{s}}\}_{\mathbf{s} \in \mathbf{S}_0}$  is a diffusion process with state space  $\mathbf{S}_0$ . Thus, it is clear that there are two completely different approaches for the construction of infinite-dimensional stochastic dynamics related to the Airy RPF with  $\beta = 2$ . We proved that these two infinite-dimensional stochastic dynamics are the same in [34].

**Theorem 2.4** ([34, Theorem 2.2]). *Assume  $\beta = 2$ . Let  $\{\mathbf{Q}_{\mathbf{s}}\}_{\mathbf{s} \in \mathbf{S}_0}$  be as above. Then,  $\{\mathbf{Q}_{\mathbf{s}}\}_{\mathbf{s} \in \mathbf{S}_0}$  satisfies (2.6)–(2.8). The associated labeled process  $\mathbf{X} = \mathbf{l}_{\text{path}}(\mathbf{X})$  is a unique strong solution of ISDE (1.9) with initial condition  $\mathbf{s} = \mathbf{l}(\mathbf{s})$  for  $\mu_{\text{Ai},2}$ -a.s.  $\mathbf{s}$ . Moreover,*

$$\mathbf{P}_{\mathbf{s}} = \mathbf{Q}_{\mathbf{s}} \quad \text{for } \mu_{\text{Ai},2}\text{-a.s. } \mathbf{s}.$$

**Remark 2.4.** *The key point of the proof of Theorem 2.4 is the uniqueness theorem Theorem 2.3 and the tail triviality of  $\mu_{\text{Ai},2}$  proved in [31]. In [31], it was proved that the tail  $\sigma$ -field of determinantal random point fields appearing random matrix theory with continuous Hermitian kernels is trivial, which contains the tail triviality of  $\mu_{\text{Ai},2}$  as a special case.*

For  $\mathbf{s} \in \mathbf{S}_0$  and  $\mathbf{n} \in \mathbb{N}$  we set  $\mathbf{l}^{\mathbf{n}}(\mathbf{s}) = (s_1, s_2, \dots, s_{\mathbf{n}}) \in \mathbb{R}^{\mathbf{n}}$  such that  $s_i \geq s_{i+1}$  for all  $i$ . Let  $\mathbf{X}_t^{\mathbf{n}} = (X_t^1, X_t^2, \dots, X_t^{\mathbf{n}})$  be the solution of (1.8) with  $\mathbf{X}_0^{\mathbf{n}} = \mathbf{l}^{\mathbf{n}}(\mathbf{s}) = (s_1, s_2, \dots, s_{\mathbf{n}})$ . Let  $\mathbf{X}^{\mathbf{n},m} = (X^{\mathbf{n},1}, \dots, X^{\mathbf{n},m})$  be the first  $m$ -components of  $\mathbf{X}^{\mathbf{n}}$ . Let  $\mu_{\text{Ai},\beta}^{\mathbf{n}}$  be the RPF whose labeled density  $m_{\text{Ai},\beta}^{\mathbf{n}}$  is given by (1.3). Then, we have the following result from [34, Corollary 2.3] immediately.

**Theorem 2.5** ([34, Corollary 2.3]). *Let  $\mathbf{P}_{\mu_{\text{Ai},2}^{\mathbf{n}}}$  be the distribution of the unlabeled process  $\{\mathbf{u}(\mathbf{X}_t^{\mathbf{n}})\}$  such that  $\mathbf{u}(\mathbf{X}_0^{\mathbf{n}}) = \mu_{\text{Ai},2}^{\mathbf{n}}$  in law.*

- (1)  $\mathbf{P}_{\mu_{\text{Ai},2}^{\mathbf{n}}}$  weakly converges to  $\mathbf{P}_{\mu_{\text{Ai},2}}$  as  $\mathbf{n} \rightarrow \infty$ .
- (2) Under  $\mathbf{P}_{\mu_{\text{Ai},2}^{\mathbf{n}}}$  and  $\mathbf{P}_{\mu_{\text{Ai},2}}$ , the first  $\mathbf{m}$ -labeled processes  $\mathbf{X}^{\mathbf{n},m}$  converge to the limit  $\mathbf{X}^{\mathbf{m}} = (X^1, \dots, X^{\mathbf{m}})$  as follows:

$$\lim_{\mathbf{n} \rightarrow \infty} \mathbf{X}^{\mathbf{n},m} = \mathbf{X}^{\mathbf{m}} \quad \text{weakly in } C([0, \infty); \mathbb{R}^{\mathbf{m}}) \text{ for each } \mathbf{m} \in \mathbb{N}.$$

Next, we turn to a Girsanov formula. For a fixed  $T \in (0, \infty)$ , we denote by  $\mathcal{W}_{\mathbf{s}}^{\mathbf{m}}$  the distribution of  $\mathbb{R}^{\mathbf{m}}$ -valued Brownian motion  $\{(B_t^i + s_i)_{i=1}^{\mathbf{m}}\}_{t \in [0, T]}$  starting at  $\mathbf{s} = (s_i) \in \mathbb{R}^{\mathbf{m}}$ ,  $\mathbf{m} \in \mathbb{N} \cup \{\infty\}$ . Let  $\mathbf{P}_{\mathbf{s}}$  be the distribution of the solution  $\mathbf{X}$  starting at  $\mathbf{s}$  as in Theorem 2.2. It is then clear that the probability measure  $\mathbf{P}_{\mathbf{s}}$  is *not* absolutely continuous with respect to  $\mathcal{W}_{\mathbf{s}}^{\infty}$ . We, therefore, formulate a Girsanov-type formula for a finite number of particles  $\mathbf{X}^{\mathbf{m}} = \{(X_t^1, \dots, X_t^{\mathbf{m}})\}_{t \in [0, T]}$  for each  $\mathbf{m} \in \mathbb{N}$ .

We set  $\mathbf{X}^{\mathbf{m}*} = \{(X_t^n)_{n=\mathbf{m}+1}^{\infty}\}_{t \in [0, T]}$ , and introduce the regular conditional probability  $\mathbf{P}_{\mathbf{s}}^{\mathbf{X}^{\mathbf{m}*}}$  of  $\mathbf{P}_{\mathbf{s}}(\mathbf{X}^{\mathbf{m}} \in \cdot)$  conditioned at  $\mathbf{X}^{\mathbf{m}*}$ :

$$\mathbf{P}_{\mathbf{s}}^{\mathbf{X}^{\mathbf{m}*}}(\cdot) = \mathbf{P}_{\mathbf{s}}(\mathbf{X}^{\mathbf{m}} \in \cdot | \mathbf{X}^{\mathbf{m}*}).$$

Let  $\mathbf{b}_{\mathbf{X}}^{\mathbf{m}}: \mathbb{R}^{\mathbf{m}} \times [0, T] \rightarrow \mathbb{R}^{\mathbf{m}}$  be the vector  $\mathbf{b}_{\mathbf{X}}^{\mathbf{m}} = (b_{\mathbf{X},i}^{\mathbf{m}})_{1 \leq i \leq \mathbf{m}}$  such that

$$b_{\mathbf{X},i}^{\mathbf{m}}(\mathbf{x}, t) = \frac{\beta}{2} \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{\mathbf{m}} \frac{1}{x_i - x_j} + \lim_{r \rightarrow \infty} \left( \sum_{\substack{j=\mathbf{m}+1 \\ |X_t^j| < r}}^{\infty} \frac{1}{x_i - X_t^j} - \int_{|y| < r} \frac{\hat{\varrho}(y)}{-y} dy \right) \right\}.$$

**Theorem 2.6.** *Assume  $\beta = 1, 2, 4$  and  $T \in (0, \infty)$ . Let  $\{\mathbf{P}_{\mathbf{s}}\}$  be as in Theorem 2.2. For  $h \in \mathbb{N}$ , let  $\tau_h: C([0, T]; \mathbb{R}^{\mathbf{m}}) \rightarrow \mathbb{R} \cup \{\infty\}$  be the stopping time with respect to the canonical filtering such that*

$$\tau_h(W) = \inf\{t \wedge T; h \leq \int_0^t |\mathbf{b}_{\mathbf{X}}^{\mathbf{m}}(W_u, u)|^2 du\}.$$

Let  $\mathbf{s} \in \mathfrak{l}(S_{\mu_{\text{Ai},\beta}})$ . Let  $\mathbf{P}_{\mathbf{s},h}^{\mathbf{X}^{m*}} = \mathbf{P}_{\mathbf{s}}^{\mathbf{X}^{m*}} \circ (W_{\cdot \wedge \tau_h})^{-1}$  and  $\mathcal{W}_{\mathbf{s},h}^m = \mathcal{W}_{\mathbf{s}}^m \circ (W_{\cdot \wedge \tau_h})^{-1}$ . Then, for  $\mathbf{P}_{\mathbf{s}}$ -a.s.  $\mathbf{X}^{m*}$ , the distribution  $\mathbf{P}_{\mathbf{s},h}^{\mathbf{X}^{m*}}$  is absolutely continuous w.r.t.  $\mathcal{W}_{\mathbf{s},h}^m$  with Radon-Nikodym density

$$(2.10) \quad \frac{d\mathbf{P}_{\mathbf{s},h}^{\mathbf{X}^{m*}}}{d\mathcal{W}_{\mathbf{s},h}^m} = e^{\int_0^{\cdot \wedge \tau_h} \mathbf{b}_{\mathbf{X}}^m(W_u, u) dW_u - \frac{1}{2} \int_0^{\cdot \wedge \tau_h} |\mathbf{b}_{\mathbf{X}}^m(W_u, u)|^2 du}.$$

Furthermore, for  $\mathbf{P}_{\mathbf{s}}$ -a.s.  $\mathbf{X}^{m*}$ ,

$$(2.11) \quad \lim_{h \rightarrow \infty} \tau_h(W) = T \text{ for } \mathbf{P}_{\mathbf{s}}^{\mathbf{X}^{m*}}\text{-a.s. } W.$$

**Remark 2.5.** (1) Theorem 2.6 implies that the local properties of tagged particles of  $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$  are the same as those of Brownian motion  $\mathbf{B} = (B^i)_{i \in \mathbb{N}}$ . In particular, each tagged particle  $X = \{X_t^i\}$  is non-differentiable and  $\alpha$ -Hölder continuous for any  $\alpha < 1/2$  in  $t$ .

(2) Since the unlabeled process  $\mathbf{X}$  is reversible, the time parameter can be extended from  $[0, \infty)$  to  $\mathbb{R}$  by the stationarity of the time shift. Assume  $\beta = 2$  and consider the process given by the extended Airy kernel. Let  $\mathcal{A}(t)$  be the top particle. Thus far we have denoted this by  $X_t^1$ .  $\mathcal{A}(t)$  is usually called the Airy process. We consider the case of a time stationary Airy process. So the distribution of  $\mathcal{A}(t)$  is independent of  $t$ . In [13, Conjecture 1.5] Johansson conjectured that  $H(t) = \mathcal{A}(t) - t^2$  almost surely has a unique maximum point in  $[-T, T]$ . We have an immediate affirmative answer for this conjecture from Theorem 2.6. In fact, this is the case of the Brownian path, and the Airy process is absolutely continuous on the time interval  $[-T, T]$  with respect to Brownian motions starting from the distribution of  $\mathcal{A}(0)$  at time  $-T$ . We remark that the conjecture referred to above has already been solved by Corwin [4] and Hägg [10] using a different method.

### 3 Preliminaries for the proof of main theorems

In this section we prepare some Lemmas for the proof of the main theorems (Theorems 2.1–2.3 and 2.6).

The key notions for the existence of  $\mu$ -reversible diffusions on configuration space  $S$  and their SDE representations are the quasi-Gibbs property and the logarithmic derivative of  $\mu$  introduced in [29] and [28], respectively, which we now explain.

Let  $S = \mathbb{R}$ . Although  $S$  was taken to be  $\mathbb{R}^d$  in [29] and [28], here we consider only  $\mathbb{R}$  but keep the notation according to [29] and [28]. We introduce a Hamiltonian on a bounded Borel set  $A$ . For Borel measurable functions  $\Phi : S \rightarrow \mathbb{R} \cup \{\infty\}$  and  $\Psi : S \times S \rightarrow \mathbb{R} \cup \{\infty\}$  with  $\Psi(x, y) = \Psi(y, x)$ , let

$$\mathcal{H}_A^{\Phi, \Psi}(\mathbf{x}) = \sum_{x_i \in A} \Phi(x_i) + \sum_{x_i, x_j \in A, i < j} \Psi(x_i, x_j), \quad \text{where } \mathbf{x} = \sum_i \delta_{x_i}.$$

We assume  $\Phi < \infty$  a.e. to avoid triviality. Functions  $\Phi$  and  $\Psi$  are called free and interaction potentials, respectively.

For two measures  $\nu_1$  and  $\nu_2$  on a measurable space  $(\Omega, \mathcal{B})$  we write  $\nu_1 \leq \nu_2$  if  $\nu_1(A) \leq \nu_2(A)$  for all  $A \in \mathcal{B}$ . We say that a sequence of finite Radon measures  $\{\mu^n\}$  on a Polish space  $\Omega$  converges weakly to a finite Radon measure  $\nu$  if  $\lim_{n \rightarrow \infty} \int f d\mu^n = \int f d\nu$  for all  $f \in C_b(\Omega)$ .

For an increasing sequence  $\{b_r\}$  of natural numbers we set

$$S_r = \{s \in S; |s| < b_r\}, \quad \mathbf{S}_r^m = \{\mathbf{s} \in \mathbf{S}; \mathbf{s}(S_r) = m\}.$$

Although  $S_r$  and  $\mathbf{S}_r^m$  depend on the sequence  $\{b_r\}$ , we omit it from the notation.

Let  $\Lambda_r$  be the Poisson RPF whose intensity is  $1_{S_r} dx$ . We set

$$\Lambda_r^m = \Lambda_r(\cdot \cap \mathbf{S}_r^m).$$

Note that  $\Lambda_r = \sum_{m=0}^{\infty} \Lambda_r^m$ .

**Definition 3.1.** A probability measure  $\mu$  is said to be a  $(\Phi, \Psi)$ -quasi Gibbs measure if there exists an increasing sequence  $\{b_r\}$  of natural numbers such that, for each  $r, m \in \mathbb{N}$ ,  $\mu_r^m := \mu(\cdot \cap \mathbf{S}_r^m)$  satisfies

$$c_1^{-1} e^{-\mathcal{H}_r(\mathbf{x})} \Lambda_r^m(d\mathbf{x}) \leq \mu_{r,\mathbf{s}}^m(d\mathbf{x}) \leq c_1 e^{-\mathcal{H}_r(\mathbf{x})} \Lambda_r^m(d\mathbf{x}) \quad \text{for } \mu_r^m\text{-a.e. } \mathbf{s} \in \mathbf{S}.$$

Here  $\mathcal{H}_r(\mathbf{x}) = \mathcal{H}_{S_r}^{\Phi, \Psi}(\mathbf{x})$ ,  $c_1 = c_1(r, m, \pi_{S_r^c}(\mathbf{s}))$  is a positive constant and  $\mu_{r,\mathbf{s}}^m$  is the regular conditional probability measure of  $\mu_r^m$  defined as

$$\mu_{r,\mathbf{s}}^m(d\mathbf{x}) = \mu_r^m(\pi_{S_r} \in d\mathbf{x} \mid \pi_{S_r^c}(\mathbf{s})).$$

**Remark 3.1.** (0) The notion of quasi-Gibbs states was introduced in previous papers of the first author [28, 29, 30], where a subordinate sequence  $\{\mu_{r,k+1}^m\}_{k \in \mathbb{N}}$  of measures was introduced to define the quasi-Gibbs state. We note that Definition 3.1 above is equivalent to this.

(1) Recall that a probability measure  $\mu$  is said to be a  $(\Phi, \Psi)$ -canonical Gibbs measure if  $\mu$  satisfies the Dobrushin-Lanford-Ruelle (DLR) equation (3.1); that is, for each  $r, m \in \mathbb{N}$ , the regular conditional probability  $\mu_{r,\mathbf{s}}^m$  satisfies

$$(3.1) \quad \mu_{r,\mathbf{s}}^m(d\mathbf{x}) = \frac{1}{c_2} e^{-\mathcal{H}_r(\mathbf{x}) - \mathcal{I}_r(\mathbf{x}, \mathbf{s})} \Lambda_r^m(d\mathbf{x}) \quad \text{for } \mu_r^m\text{-a.e. } \mathbf{s}.$$

Here  $0 < c_2 < \infty$  is the normalization and, for  $\mathbf{x} = \sum_i \delta_{x_i}$  and  $\mathbf{s} = \sum_j \delta_{s_j}$ , we set

$$\mathcal{I}_r(\mathbf{x}, \mathbf{s}) = \sum_{x_i \in S_r, s_j \in S_r^c} \Psi(x_i, s_j).$$

By construction  $(\Phi, \Psi)$ -canonical Gibbs measures are  $(\Phi, \Psi)$ -quasi Gibbs measures. The converse is, however, not true. This is a common feature of infinite volume RPFs appearing in random matrix theory such as the Ginibre RPF and  $\text{sine}_\beta$  RPF ( $\beta = 1, 2, 4$ ). In fact, when  $\Psi(x, y) = -\beta \log |x - y|$  and  $\mu$  is translation invariant,  $\mu$  is not a  $(\Phi, \Psi)$ -canonical Gibbs measure because the DLR equation does not make sense. Indeed,  $|\mathcal{I}_r(\mathbf{x}, \mathbf{s})| = \infty$  for  $\mu$ -a.s.  $\mathbf{s}$ . The point is that a cancellation can be expected between  $c_2$  and  $e^{-\mathcal{I}_r(\mathbf{x}, \mathbf{s})}$  even if  $|\mathcal{I}_r(\mathbf{x}, \mathbf{s})| = \infty$ . The main task of the proof of the quasi-Gibbs property of Airy RPFs is to find this cancellation.

(2) Unlike canonical Gibbs measures, the notion of quasi-Gibbs measures is quite flexible for free potentials. Indeed, if  $\mu$  is a  $(\Phi, \Psi)$ -quasi Gibbs measure, then  $\mu$  is also a  $(\Phi + F, \Psi)$ -quasi Gibbs measure for any locally bounded measurable function  $F$ . So we write  $\mu$  a  $\Psi$ -quasi Gibbs measure, if  $\mu$  is a  $(0, \Psi)$ -quasi Gibbs measure.

The significance of the quasi-Gibbs property is that, combined with the local boundedness of the correlation functions and minimal regularity of potentials of a given RPF  $\mu$ , it yields the construction of diffusions associated with the RPF  $\mu$ . This result is a consequence of the general theory in [29], [28], and [27], involving Dirichlet forms. We quote some of these in Lemma 3.1 and Lemma 3.2.

Let  $\mu$  be an RPF over  $S$  with correlation functions  $\rho^k$  ( $k \in \mathbb{N}$ ). Let  $\sigma_r^k$  be the  $k$ -density function of  $\mu$  with respect to the Lebesgue measure on  $S_r^k$ . We assume:

(A.3.1)  $\rho^1 \in L^1_{\text{loc}}(S, dx)$  and  $\sigma_r^k \in L^2(S_r^k, d\mathbf{x}_k)$  for all  $k, r \in \mathbb{N}$ .

(A.3.2)  $\mu$  is a  $(\Phi, \Psi)$ -quasi Gibbs measure with upper semi-continuous  $(\Phi, \Psi)$ .

We quote a result from [29] and [25] with minor generalization.

**Lemma 3.1.** *Assume (A.3.1) and (A.3.2). Then, the following holds.*

(1)  $(\mathcal{E}^\mu, \mathcal{D}_0^\mu)$  is closable on  $L^2(S, \mu)$ .

(2) There exists a  $\mu$ -reversible diffusion  $\{\mathbf{P}_s\}_{s \in S_\mu}$  associated with  $(\mathcal{E}^\mu, \mathcal{D}^\mu)$  on  $L^2(S, \mu)$ . Here  $(\mathcal{E}^\mu, \mathcal{D}^\mu)$  is the closure of  $(\mathcal{E}^\mu, \mathcal{D}_0^\mu)$  on  $L^2(S, \mu)$ .

*Proof.* The claim (1) follows from [29]. If we replace  $\sigma_r^k \in L^2(S_r^k, d\mathbf{x}_k)$  by  $\sigma_r^k \in L^\infty(S_r^k, d\mathbf{x}_k)$  in (A.3.1), then the claim (2) was proved in [25, Theorem 2.1]. The assumption  $\sigma_r^k \in L^\infty(S_r^k, d\mathbf{x}_k)$  is used only in the proof of [25, Lemma 2.4]. We can still prove [25, Lemma 2.4] under the present assumption by using the Schwarz inequality in the second line in [25, 126 p].  $\square$

We use Lemma 3.1 to prove Theorem 2.1. From Lemma 3.1 we have unlabeled dynamics  $\mathbf{X}_t = \sum_i \delta_{X_t^i}$ . Under additional assumptions, we can construct labeled dynamics  $\mathbf{X} = \mathbf{l}_{\text{path}}(\mathbf{X}) = \{(X_t^i)_{i \in \mathbb{N}}\}_{t \in I_i}$  (see (A.3.3), (A.3.4), and (3.5) below). Next, we quote another general result concerning the ISDE representation of the labeled dynamics  $\mathbf{X}$ . For this we recall the notion of reduced Palm and Campbell measures.



For a RPF  $\mu$  over  $S$ , probability measure  $\mu_{\mathbf{x}}$  is called the reduced Palm measure of  $\mu$  conditioned at  $\mathbf{x} = (x_1, \dots, x_k) \in S^k$  if it is defined as

$$\mu_{\mathbf{x}} = \mu(\cdot - \sum_{i=1}^k \delta_{x_i} \mid s(x_i) \geq 1 \text{ for } i = 1, \dots, k).$$

Let  $\rho^k$  be the  $k$ -point correlation function of  $\mu$  w.r.t. the Lebesgue measure. Let  $S^{[k]} = S^k \times S$  and  $\mu^{[k]}$  be the measure on  $S^{[k]}$  defined as

$$\mu^{[k]}(A \times B) = \int_A \mu_{\mathbf{x}}(B) \rho^k(\mathbf{x}) d\mathbf{x}.$$

Here we set  $d\mathbf{x} = dx_1 \cdots dx_k$  for  $\mathbf{x} = (x_1, \dots, x_k) \in S^k$ . The measure  $\mu^{[k]}$  is called the  $k$ -Campbell measure. By convention we set  $\mu^{[0]} = \mu$ .

Let  $\text{Cap}^\mu$  be the capacity associated the Dirichlet space  $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(S, \mu))$ . We assume:

(A.3.3)  $\text{Cap}^\mu(\{S_{\text{s.i.}}\}^c) = 0$ .

(A.3.4) There exists a  $T > 0$  such that for each  $R > 0$ ,

$$(3.2) \quad \liminf_{r \rightarrow \infty} \mathcal{N}\left(\frac{r}{\sqrt{(r+R)T}}\right) \left\{ \int_{|x| \leq r+R} \rho^1(x) dx \right\} = 0.$$

Here  $\mathcal{N}(t) = \int_t^\infty (1/\sqrt{2\pi}) e^{-|x|^2/2} dx$ .

Assumptions (A.3.3) and (A.3.4) have clear dynamical interpretations. Indeed, (A.3.3) means that the particles never collide with one another:

$$(3.3) \quad \mathbb{P}_{\mathbf{s}}(\mathbf{X}_t \in S_{\text{s.i.}} \quad \text{for all } 0 \leq t < \infty) = 1 \quad \text{for q.e. } \mathbf{s} \in S.$$

Moreover, we deduce from (A.3.4) that no labeled particle ever explodes:

$$(3.4) \quad \mathbb{P}_{\mathbf{s}}\left(\sup_{t \in [0, T]} |X_t^i| < \infty \quad \text{for all } T, i \in \mathbb{N}\right) = 1 \quad \text{for q.e. } \mathbf{s} \in S.$$

Here q.e. means quasi-everywhere (see [9]). All subsequent arguments follow from (3.3) and (3.4) instead of (A.3.3) and (A.3.4). Here we state the assumption as geometrically as possible. We remark that (A.3.3) is equivalent to (3.3), and that (A.3.4) is a sufficient condition for (3.4).

From (3.3) and (3.4) we can and do take a state space  $S_\mu$  of the unlabeled diffusion  $(X, P_{\mathbf{s}})$  in such a way that

$$S_\mu \subset S_{\text{s.i.}}$$

and that the equations in (3.3) and (3.4) holds for all  $\mathbf{s} \in S_\mu$ .

We can label the particles in such a way that  $\mathbf{X}_t = \sum_i \delta_{X_t^i}$ , where each  $\{X_t^i\}$  is a continuous process defined on the time interval  $[0, \infty)$ . This expression is unique up to the initial labeling  $\mathfrak{l}(\mathbf{X}_0) = (X_0^i)_{i \in \mathbb{N}}$  of the processes  $\{(X_t^i)_{i \in \mathbb{N}}\}_{[0, \infty)}$ . In fact, once  $\mathfrak{l}(\mathbf{X}_0) = (X_0^i)_{i \in \mathbb{N}}$  is given, each particle carries its initial label continuously by (3.3) and (3.4). This correspondence is called a label path map, and is denoted by

$$(3.5) \quad \mathfrak{l}_{\text{path}}(\mathbf{X}) = \mathbf{X}.$$

To derive the ISDE satisfied by the labeled dynamics  $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ , we introduce the notion of a logarithmic derivative of RPF  $\mu$ .

**Definition 3.2.** We call  $\mathbf{d}^\mu = (\mathbf{d}_m^\mu)_{m=1, \dots, d} \in \{L_{\text{loc}}^1(\mu^{[1]})\}^d$  a logarithmic derivative of  $\mu$  if  $\mathbf{d}^\mu$  satisfies

$$(3.6) \quad \int_{S \times S} \mathbf{d}^\mu f d\mu^{[1]} = - \int_{S \times S} \nabla_x f d\mu^{[1]} \quad \text{for all } f \in C_0^\infty(S) \otimes \mathcal{D}_\circ.$$

Very loosely, (3.6) can be written as  $\mathbf{d}^\mu = \nabla_x \log \mu^{[1]}$ . This expression is the reason that we call  $\mathbf{d}^\mu$  the logarithmic derivative of  $\mu$ . We remark that  $\mathbf{d}^\mu$  is a logarithmic derivative of the one-Campbell measure  $\mu^{[1]}$  rather than of  $\mu$ . This choice is suitable for the description of the ISDE for  $\mathbf{X}$ . Indeed,  $\frac{1}{2}\mathbf{d}^\mu$  expresses the force effected on each tagged particle  $X_t^i$  by all other infinitely many particles  $\sum_{j \neq i} \delta_{X_t^j}$ . When we deal with systems consisting of particles of  $k$  species, we can define the logarithmic derivative of the  $k$ -Campbell measures to derive the associated ISDEs. We assume:

(A.3.5) There exists a logarithmic derivative  $\mathbf{d}^\mu$  in the sense of (3.6).

The following lemma is a special case of [28, Theorem 26] with a slight modification, and is used in the proof of Theorem 2.2.

**Lemma 3.2.** Assume (A.3.1)–(A.3.5). Let  $\{\mathbf{P}_s\}_{s \in S_\mu}$  be as in Lemma 3.1. Let  $\mathfrak{l}$  be a label. Then, there exists an  $S_0$  such that

$$(3.7) \quad \mu(S_0) = 1, \quad S_0 \subset S_\mu \subset S_{\text{s.i.}},$$

and for all  $\mathbf{s} = \mathfrak{l}(s) \in \mathfrak{l}(S_0)$ , the labeled path  $\mathbf{X} = \mathfrak{l}_{\text{path}}(\mathbf{X})$  under  $\mathbf{P}_s \circ \mathfrak{l}^{-1}$  satisfies

$$(3.8) \quad dX_t^i = dB_t^i + \frac{1}{2} \mathbf{d}^\mu(X_t^i, \mathbf{X}_t^{i\Diamond}) dt \quad (i \in \mathbb{N}),$$

$$(3.9) \quad \mathbf{X}_0 = \mathbf{s}.$$

Here  $\mathbf{B} = (B^i)_{i \in \mathbb{N}}$  is an  $\mathbb{R}^\mathbb{N}$ -valued Brownian motion, and  $\mathbf{X}_t^{i\Diamond} = \sum_{j \neq i} \delta_{X_t^j}$  and  $\mathbf{s} = (s_i)_{i \in \mathbb{N}}$ . Moreover,  $\mathbf{X}$  satisfies

$$(3.10) \quad P(\mathbf{u}(\mathbf{X}_t) \in S_0, 0 \leq \forall t < \infty) = 1.$$

Here  $\mathbf{u}: S^\mathbb{N} \rightarrow S$  is the unlabeled map such that  $\mathbf{u}((x_i)) = \sum_i \delta_{x_i}$ .

Let  $\rho^1$  and  $\mathcal{N}(t) = \int_t^\infty (1/\sqrt{2\pi})e^{-|x|^2/2}dx$  be as in (3.2). Then, we assume:

(A.3.6) For each  $r, T \in \mathbb{N}$ ,

$$\int_{\mathbb{R}} \mathcal{N}\left(\frac{|x| - r}{\sqrt{T}}\right) \rho^1(x) dx < \infty.$$

For a subset  $A \subset S$  we set  $A^{[1]} = (\mathbf{u}^1)^{-1}(A) \subset S \times S$ , where  $\mathbf{u}^1(x, y) = \delta_x + y$ . To construct a unique strong solution of (3.8), we assume that the one labeled solution  $(X_t^i, \mathbf{X}_t^{i\Diamond})$  remains with set  $H^{[1]}$  for all  $t$  and that set  $H^{[1]}$  satisfies coefficient  $\mathbf{d}^\mu(x, \mathbf{s})$  taking a finite value. That is, we assume:

(A.3.7) There exists  $H \in \mathcal{B}(S)$  such that  $\mathbf{d}^\mu(x, \mathbf{s})$  takes a finite value for all  $(x, \mathbf{s}) \in H^{[1]}$  and that  $H \subset S_0$  and

$$(3.11) \quad \text{Cap}^\mu(H^c) = 0.$$

Let  $\mu^{[1]}$  be the one-Campbell measure of  $\mu$  as before. In [32] we prove that the one-labeled process  $(X_t^i, \mathbf{X}_t^{i\Diamond})$  is a  $\mu^{[1]}$ -symmetric diffusion and from (3.11), we can deduce that

$$(3.12) \quad \text{Cap}^{\mu^{[1]}}((H^{[1]})^c) = 0,$$

where  $\text{Cap}^{\mu^{[1]}}$  is the capacity of the one-labeled diffusion  $(X_t^i, \mathbf{X}_t^{i\Diamond})$ . We refer to [27, 28] for the associated Dirichlet forms which define the capacity  $\text{Cap}^{\mu^{[1]}}$ .

Next we introduce a system of finite-dimensional SDEs associated with ISDEs (3.8) and (3.9). For this we prepare a set of notations.

For a path  $\mathbf{X} = (X_t^i)_{i \in \mathbb{N}} \in C([0, \infty); \mathbb{R}^{\mathbb{N}})$  and  $m \in \mathbb{N}$ , we set  $\mathbf{X}^{m*} = \sum_{i=m+1}^\infty \delta_{X_t^i}$  and  $\mathbf{X}^{m*} = (X_t^n)_{n=m+1}^\infty$ . Let  $S_0$  be as in (3.7),  $\mathbf{S}_0 = \mathbf{l}(S_0)$ , and

$$W_{\text{sol}} = \mathbf{l}_{\text{path}}(C([0, \infty); H)).$$

Then for  $\mathbf{X} \in W_{\text{sol}}$ ,  $\mathbf{s} = (s_i)_{i=1}^\infty \in \mathbf{S}_0$ , and  $m \in \mathbb{N}$ , we introduce the finite-dimensional SDE (3.13) of  $\mathbf{Y}^m = (Y^{m,1}, \dots, Y^{m,m})$  such that

$$(3.13) \quad \begin{aligned} dY_t^{m,i} &= dB_t^i + \frac{1}{2} \mathbf{d}^\mu(Y_t^{m,i}, \mathbf{Y}_t^{m,i\Diamond} + \mathbf{X}_t^{m*}) dt \quad (i = 1, \dots, m) \\ \mathbf{Y}_0^m &= (s_1, \dots, s_m) \in S^m. \end{aligned}$$

Here we set  $\mathbf{Y}_t^{m,i\Diamond} = \sum_{j \neq i}^m \delta_{Y_t^{m,j}}$  and  $(s_1, \dots, s_m)$  are the first  $m$ -components of  $\mathbf{s}$ .

We interpret  $\mathbf{X}^{m*}$  as part of the coefficients of SDE (3.13). We assume:

(A.3.8) For each  $\mathbf{s} \in \mathbf{S}_0$  and  $\mathbf{X} \in W_{\text{sol}}$  such that  $\mathbf{X}_0 = \mathbf{s}$ , SDE (3.13) has a unique, strong solution  $\mathbf{Y}^m$  for each  $m \in \mathbb{N}$ . Moreover,  $\mathbf{Y}^m$  satisfies

$$(3.14) \quad (\mathbf{Y}^m, \mathbf{X}^{m*}) \in W_{\text{sol}}.$$

Let  $\mathcal{T}(\mathbf{S})$  be the tail  $\sigma$ -field of  $\mathbf{S}$  defined as

$$(3.15) \quad \mathcal{T}(\mathbf{S}) = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}].$$

We say  $\mu$  is tail trivial if  $\mu(A) \in \{0, 1\}$  for all  $A \in \mathcal{T}(\mathbf{S})$ . We assume:

**(A.3.9)**  $\mu$  is tail trivial.

In [32], we clarify the significance of the tail triviality to the construction of the unique, strong solution of ISDE (3.8), and we quote one of the main theorems in [32] for the proof of Theorem 2.3. The next result is a special case of [32, Theorem 2.1].

**Lemma 3.3** ([32, Theorem 2.1]). *Assume that  $\mu$  satisfies (A.3.1)–(A.3.9). Then, there exists a set  $\mathbf{S}_1$  satisfying  $\mu(\mathbf{S}_1) = 1$ ,  $\mathbf{S}_1 \subset \mathbf{S}_0$ , and the following:*

- (1) *ISDE (3.8)–(3.10) has a strong solution  $(\mathbf{X}, \mathbf{P}_s)$  for each  $s \in \mathbf{S}_1$  such that  $\{(\mathbf{X}, \mathbf{P}_s)\}_{s \in \mathbf{S}_1}$  is an  $\mathbf{S}_1$ -valued diffusion. Each associated unlabeled process  $\{(\mathbf{X}, \mathbf{P}_s)\}_{s \in \mathbf{S}_1}$  is an  $\mathbf{S}_1$ -valued,  $\mu$ -reversible diffusion. Here  $\mathbf{S}_1 = \mathbf{u}(\mathbf{S}_1)$ .*
- (2) *The family of strong solutions  $\{(\mathbf{X}, \mathbf{P}_s)\}_{s \in \mathbf{S}_1}$  of ISDE (3.8)–(3.10) satisfying the  $\mu$ -absolute continuity condition (2.9) is unique for  $\mu^l$ -a.s.  $s$ .*

*Proof.* We denote by (A1)–(A9) the conditions in [32, Theorem 2.1]. We check that conditions (A1)–(A9) in [32] are satisfied.

Assumptions (A1)–(A6) follow from (A.3.5), (A.3.2), (A.3.1), (A.3.3), (A.3.4), and (A.3.7), respectively. From (A.3.6) we can apply [32, Lemma 8.4] to obtain (A8). Assumptions (A7) and (A9) follow from (A.3.8) and (A.3.9), respectively.  $\square$

It is known in [31] that all determinantal RPFs are tail trivial. Hence, we can apply Lemma 3.3 to the case where  $\beta = 2$ . Given that in general, quasi-Gibbs measures are not tail trivial, we introduce the tail decomposition (3.16) of  $\mu$  in the following.

Let  $\mu_t, t \in \mathbf{S}$  be the regular conditional probability defined as

$$\mu_t(\cdot) = \mu(\cdot | \mathcal{T}(\mathbf{S}))(\mathbf{t}).$$

Then,  $\mu_t(A)$  is  $\mathcal{T}(\mathbf{S})$ -measurable for any  $A \in \mathcal{B}(\mathbf{S})$  by definition, and we have

$$(3.16) \quad \mu(\cdot) = \int_{\mathbf{S}} \mu_t(\cdot) \mu(d\mathbf{t}).$$

We quote:

**Lemma 3.4** ([32, Lemma 11.2]). *Assume that  $\mu$  is a quasi-Gibbs measure. Then, there exists a subset  $H_0$  of  $H$  satisfying that  $\mu(H_0) = 1$  and for all  $\mathbf{t}, \mathbf{u} \in H_0$*

$$(3.17) \quad \mu_{\mathbf{t}}(\mathbf{A}) \in \{0, 1\} \quad \text{for all } \mathbf{A} \in \mathcal{T}(\mathbf{S}).$$

$$(3.18) \quad \mu_{\mathbf{t}}(\{\mathbf{s} \in \mathbf{S} : \mu_{\mathbf{s}} = \mu_{\mathbf{t}}\}) = 1.$$

$$(3.19) \quad \mu_{\mathbf{t}} \text{ and } \mu_{\mathbf{u}} \text{ are mutually singular on } \mathcal{T}(\mathbf{S}) \text{ if } \mu_{\mathbf{t}} \neq \mu_{\mathbf{u}}.$$

From (3.17)–(3.19), we introduce the equivalent relation denoted by  $\sim_{\text{Tail}}$  such that  $\mathbf{t} \sim_{\text{Tail}} \mathbf{u}$  if and only if  $\mu_{\mathbf{t}} = \mu_{\mathbf{u}}$ . We denote by  $\mathbf{S}/\mathcal{T}(\mathbf{S})$  the quotient space given by  $\sim_{\text{Tail}}$ .

A significant property of the tail decomposition (3.16) is the stability of assumptions (A.3.1)–(A.3.8) as seen in [32, Lemma 7.2]. Combining this with Lemma 3.3, we can dispense with (A.3.9) as follows. The next lemma is a special case of [32, Theorem 2.2].

**Lemma 3.5** ([32, Theorem 2.2]). *Assume that  $\mu$  satisfies (A.3.1)–(A.3.8). Let  $\mathbf{l}$  be a label. Then, there exists  $\mathbf{S}_2$  satisfying  $\mu(\mathbf{S}_2) = 1$ ,  $\mathbf{S}_2 \subset \mathbf{S}_1$ , and the following:*

- (1) *ISDE (3.8)–(3.10) has a strong solution  $(\mathbf{X}, \mathbf{P}_{\mathbf{s}})$  for each  $\mathbf{s} \in \mathbf{S}_2 = \mathbf{l}(\mathbf{S}_2)$ .*
- (2)  *$\mathbf{S}_2$  can be decomposed as a disjoint sum  $\mathbf{S}_2 = \sum_{\mathbf{t} \in \mathbf{S}/\mathcal{T}(\mathbf{S})} \mathbf{S}_{2,\mathbf{t}}$ , such that*

$$\mu_{\mathbf{t}}(\mathbf{S}_{2,\mathbf{t}}) = 1, \text{ where } \mathbf{S}_{2,\mathbf{t}} = \mathbf{u}(\mathbf{S}_{2,\mathbf{t}}),$$

*and the sub collection  $\{(\mathbf{X}, \mathbf{P}_{\mathbf{s}})\}_{\mathbf{s} \in \mathbf{S}_{2,\mathbf{t}}}$  is an  $\mathbf{S}_{2,\mathbf{t}}$ -valued,  $\mu_{\mathbf{t}}$ -reversible diffusion for  $\mu$ -a.s.  $\mathbf{t}$ . Here  $\mathbf{P}_{\mathbf{s}} = \mathbf{P}_{\mathbf{s}} \circ \mathbf{u}^{-1}$  and  $\mathbf{s} = \mathbf{l}(\mathbf{s})$ . Moreover,  $\{(\mathbf{X}, \mathbf{P}_{\mathbf{s}})\}_{\mathbf{s} \in \mathbf{S}_{2,\mathbf{t}}}$  is an  $\mathbf{S}_{2,\mathbf{t}}$ -valued diffusion for  $\mu$ -a.s.  $\mathbf{t}$ .*

- (3) *The family of solutions  $\{(\mathbf{X}, \mathbf{P}_{\mathbf{s}})\}_{\mathbf{s} \in \mathbf{S}_2}$  of ISDE (3.8)–(3.10) satisfying*

$$(3.20) \quad \mathbf{P}_{\mu_{\mathbf{t}}} \circ \mathbf{X}_{\mathbf{u}}^{-1} \prec \mu_{\mathbf{t}} \quad \text{for all } \mathbf{u} \in [0, T].$$

*for  $\mu$ -a.s.  $\mathbf{t}$ , is pathwise unique for  $\mu^{\mathbf{l}}$ -a.s.  $\mathbf{s}$ , and becomes a family of strong solutions.*

**Remark 3.2.** (1) *Lemma 3.5 (1) asserts the strong uniqueness in the sense that a family of solutions with (3.20) automatically becomes a family of strong solutions that satisfies pathwise uniqueness under (3.20).*

(2) *The uniqueness in Lemma 3.5 does not exclude the possibility of the existence of solutions not satisfying (3.20).*

(3) *The diffusion  $\{(\mathbf{X}, \mathbf{P}_{\mathbf{s}})\}_{\mathbf{s} \in \mathbf{S}_{2,\mathbf{t}}}$  in Lemma 3.5 conserves the tail  $\sigma$ -field of  $\mu$ .*

## 4 Finite particle approximation of Airy RPFs

We use assumptions (A.3.1)–(A.3.5) given in Section 3 to prove Theorem 2.1 and Theorem 2.2 in Subsection 5.1 and Subsection 5.2, respectively. In this section we

check the assumptions (A.3.1), (A.3.3), and (A.3.4). We also investigate the property of correlation functions of  $n$ -particle approximation of Airy RPFs, and prove Proposition 4.6, which is a key estimate of the derivation of logarithmic derivative of Airy RPFs. To prove Proposition 4.6 we use Lemma 4.5, whose proof is long and so postponed to Subsection 6.5.

**Lemma 4.1.** *Let  $\beta = 1, 2, 4$ . Then, the following holds.*

- (1)  $\mu_{\text{Ai},\beta}$  satisfies (A.3.1) and (A.3.4).
- (2) Assume that  $\mu_{\text{Ai},\beta}$  satisfies (A.3.2). Then  $\mu_{\text{Ai},\beta}$  satisfies (A.3.3).

*Proof.* Recall that correlation functions  $\rho_{\text{Ai},\beta}^k(\mathbf{x}_k)$  are given by the determinant with elements  $K_{\text{Ai},2}(x_i, x_j)$  if  $\beta = 2$ , and the quaternion determinant with elements  $K_{\text{Ai},\beta}(x_i, x_j)$  if  $\beta = 1, 4$ . We deduce (A.3.1) and (A.3.4) from the local boundedness of kernels  $K_{\text{Ai},\beta}(x, y)$  and the asymptote of  $\rho_{\text{Ai},\beta}^1(x) = \mathcal{O}(|x|^{1/2})$ .

We deduce (A.3.3) from [26, Theorem 2.1] and the local Lipschitz continuity of kernel  $K_{\text{Ai},\beta}$ . This completes the proof.  $\square$

From Lemma 3.1, Lemma 3.2, and Lemma 4.1, it only remains to prove (A.3.2) and (A.3.5) with the logarithm derivative given by (5.15) for proving Theorem 2.1 and Theorem 2.2. Hence, we study the quasi-Gibbs property and logarithmic derivative of  $\mu_{\text{Ai},\beta}$  in Subsection 5.1 and Subsection 5.2, respectively. In the rest of this section, we collect some estimates used in these sections.

Let  $\mu_{\text{Ai},\beta}^n$  be the RPF whose labeled density  $m_{\text{Ai},\beta}^n$  is given by (1.3) as before. The sequence  $\{\mu_{\text{Ai},\beta}^n\}_{n \in \mathbb{N}}$  of probability measures on the space of unlabeled finite particles, which approximates measure  $\mu_{\text{Ai},\beta}$ , plays an important role in the following two sections.

The  $n$ -correlation function of  $\mu_{\text{Ai},\beta}^n$  is denoted by  $\rho_{\text{Ai},\beta}^{n,n}$ . For  $\beta = 2$ ,  $\rho_{\text{Ai},2}^{n,n}$  is represented by the determinant with correlation kernel  $K_{\text{Ai},2}^n$  defined by (6.21), and for  $\beta = 1, 4$ ,  $\rho_{\text{Ai},\beta}^{n,n}$  is expressed as

$$(4.1) \quad \rho_{\text{Ai},\beta}^{n,n}(x_1, \dots, x_n) = \text{qdet}[K_{\text{Ai},\beta}^n(x_i, x_j)]_{i,j=1,\dots,n}$$

with quaternion-valued correlation kernels  $K_{\text{Ai},\beta}^n$  defined by (6.22). (See for example, [20, 1, 6].) The reduced Palm measure is also a (quaternion) determinantal point process with its kernel is expressed as

$$(4.2) \quad K_{\text{Ai},\beta,x}^n(y, z) = K_{\text{Ai},\beta}^n(y, z) - \frac{K_{\text{Ai},\beta}^n(y, x)K_{\text{Ai},\beta}^n(x, z)}{\rho_{\text{Ai},\beta}^{n,1}(x)}$$

We refer to [37] for the proof of (4.2) in the case of  $\beta = 2$ . The case  $\beta = 1, 4$  can be proved similarly. We also remark that the one-correlation functions of the reduced Palm measures are given by

$$(4.3) \quad \rho_{\text{Ai},\beta,x}^{n,1}(y) = \frac{\rho_{\text{Ai},\beta}^{n,2}(x, y)}{\rho_{\text{Ai},\beta}^{n,1}(x)}.$$

**Lemma 4.2.** *Let  $\beta = 1, 2, 4$ . Then, the following holds.*

$$(4.4) \quad \lim_{n \rightarrow \infty} \rho_{\text{Ai}, \beta}^{n, n} = \rho_{\text{Ai}, \beta}^n, \quad \lim_{n \rightarrow \infty} \partial_i \rho_{\text{Ai}, \beta}^{n, n} = \partial_i \rho_{\text{Ai}, \beta}^n \quad \text{compact uniformly,}$$

$$(4.5) \quad \lim_{n \rightarrow \infty} \rho_{\text{Ai}, \beta, x}^{n, 1} = \rho_{\text{Ai}, \beta, x}^1, \quad \lim_{n \rightarrow \infty} \partial_1 \rho_{\text{Ai}, \beta, x}^{n, 1} = \partial_1 \rho_{\text{Ai}, \beta, x}^1$$

*compact uniformly.*

Here  $\partial_i$  is the partial derivative in the  $i$ -th variable, where  $i = 1, \dots, n$ .

*Proof.* Recall that  $K_{\text{Ai}, 2}^n(x, y) = K_{\text{Ai}, 2}^n(y, x)$ . It is well known that  $K_{\text{Ai}, 2}^n$  and  $\partial_x K_{\text{Ai}, 2}^n$  converge to  $K_{\text{Ai}, 2}$  and  $\partial_x K_{\text{Ai}, 2}$  compact uniformly. The same convergence also holds for  $\beta = 1, 4$  from the above combined with the definitions of  $K_{\text{Ai}, \beta}$  and  $K_{\text{Ai}, \beta}^n$  for  $\beta = 1, 4$  given by (6.3) and (6.22). Hence we obtain (4.4) from the definition of correlation functions given by (2.1) and (4.1).

We deduce (4.5) directly from (4.3), (4.4), and  $\rho_{\text{Ai}, \beta}(x) > 0$ .  $\square$

Next, we quote an estimate from [17] and give its consequence. Let  $\hat{\varrho}^n$  and  $\hat{\varrho}$  be as in (1.13) and (1.10), respectively. We take these functions  $\hat{\varrho}^n$  and  $\hat{\varrho}$  as the main terms of the approximations for the one-correlation functions  $\rho_{\text{Ai}, \beta}^{n, 1}$  and  $\rho_{\text{Ai}, \beta}^1$ , respectively.

**Lemma 4.3.** *Let  $\beta = 1, 2, 4$ . Then, there exists a positive constant  $c_3$  such that*

$$(4.6) \quad |\rho_{\text{Ai}, \beta}^{n, 1}(x) - \hat{\varrho}^n(x)| \leq c_3 \left\{ \frac{1}{|x|} + \frac{\mathbf{1}(\beta \neq 2)}{|x|^{1/4}} \right\}$$

*for all  $x \in [-2n^{2/3}, \infty)$ ,  $n \in \mathbb{N}$ ,*

$$(4.7) \quad |\rho_{\text{Ai}, \beta}^1(x) - \hat{\varrho}(x)| \leq c_3 \left\{ \frac{1}{|x|} + \frac{\mathbf{1}(\beta = 4)}{|x|^{1/4}} \right\} \quad \text{for all } x \in \mathbb{R}.$$

Here  $\mathbf{1}(\beta \neq 2) = 1$  if  $\beta \neq 2$ , and  $\mathbf{1}(\beta \neq 2) = 0$  otherwise;  $\mathbf{1}(\beta = 4)$  is defined similarly. Furthermore,

$$(4.8) \quad \rho_{\text{Ai}, \beta}^{n, 1}(x) \leq c_3 \sqrt{|x| + 1} \quad \text{for all } x \in \mathbb{R}, \quad n \in \mathbb{N},$$

$$(4.9) \quad \rho_{\text{Ai}, \beta}^1(x) \leq c_3 \sqrt{|x| + 1} \quad \text{for all } x \in \mathbb{R}.$$

*Proof.* If  $\beta = 2$ , then Lemma 4.3 follows from [17, Lemma 4.3]. If  $\beta = 1, 4$ , then we deduce from (4.1), (6.22) and (6.23) that the following relation holds.

$$\rho_{\text{Ai}, 1}^{n, 1}(x) = \rho_{\text{Ai}, 2}^{n, 1}(x) + \frac{1}{2} \psi_{n-1}(x) \varepsilon \psi_n(x), \quad n \in 2\mathbb{N},$$

$$\rho_{\text{Ai}, 4}^{n, 1}(x) = \frac{1}{2^{1/3}} \rho_{\text{Ai}, 2}^{n, 1}(2^{2/3}x) + \frac{\sqrt{2n+1}}{2^{17/6} n^{1/2}} \psi_{2n}(2^{2/3}x) \varepsilon \psi_{2n+1}(2^{2/3}x), \quad n \in \mathbb{N},$$

with function  $\psi_n$  and  $\varepsilon\psi_n$  defined as (6.16) and (6.24), respectively. Hence, (4.6) follows from the case where  $\beta = 2$  by using Lemma 6.9 and Lemma 6.10.

Correlation functions  $\rho_{\text{Ai},\beta}^n$ ,  $\beta = 1, 4$ , are represented by a quaternion determinant with kernels in (6.3). In particular, we deduce from (6.2) that

$$\begin{aligned}\rho_{\text{Ai},1}^1(x) &= \rho_{\text{Ai},2}^1(x) + \frac{1}{2}\text{Ai}(x)(1 - \int_x^\infty \text{Ai}(u)du) \\ \rho_{\text{Ai},4}^1(x) &= \frac{1}{2^{1/3}}\rho_{\text{Ai},2}^1(2^{2/3}x) - \frac{1}{2^{2/3}}\text{Ai}(2^{2/3}x) \int_x^\infty \text{Ai}(2^{2/3}u)du.\end{aligned}$$

Hence, (4.7) follows from the case  $\beta = 2$  with Lemmas 6.1 and 6.2.

Eq. (4.8) for  $x \in [-2n^{2/3}, \infty)$  follows from (4.6) and (1.13) immediately. Because of the symmetry of  $\rho_{\text{Ai},\beta}^{n,1}(x)$  around  $x = -n^{2/3}$ , we deduce (4.8) for all  $x \in \mathbb{R}$ .

Eq. (4.9) follows from (1.10), (4.7), and the local boundedness of  $\rho_{\text{Ai},\beta}^1$ .  $\square$

**Lemma 4.4.** *Let  $\hat{\varrho}^n$  and  $\hat{\varrho}$  be as in Lemma 4.3. Then for each  $r \in \mathbb{N}$*

$$(4.10) \quad \lim_{n \rightarrow \infty} \limsup_{s \rightarrow \infty} \sup_{|x| \leq r} \left| \int_{|y| < s} \frac{\rho_{\text{Ai},\beta,x}^{n,1}(y) - \rho_{\text{Ai},\beta,x}^1(y)}{x - y} - \frac{\hat{\varrho}^n(y) - \hat{\varrho}(y)}{x - y} dy \right| = 0,$$

$$(4.11) \quad \lim_{n \rightarrow \infty} \limsup_{s \rightarrow \infty} \sup_{|x| \leq r} \left| \int_{|y| < s} (\hat{\varrho}^n(y) - \hat{\varrho}(y)) \left\{ \frac{1}{x - y} - \frac{1}{-y} \right\} dy \right| = 0.$$

Here the integral at  $x$  and the origin in (4.11) are Cauchy's principal values.

*Proof.* From Lemma 4.2, Lemma 6.5, and Lemma 6.14 we deduce that

$$(4.12) \quad \lim_{n \rightarrow \infty} \limsup_{s \rightarrow \infty} \sup_{|x| \leq r} \left| \int_{|y| < s} \frac{\rho_{\text{Ai},\beta,x}^{n,1}(y) - \rho_{\text{Ai},\beta,x}^1(y)}{x - y} - \frac{\rho_{\text{Ai},\beta}^{n,1}(y) - \rho_{\text{Ai},\beta}^1(y)}{x - y} dy \right| = 0.$$

We deduce from Lemma 4.2 that for each  $l \in \mathbb{N}$

$$(4.13) \quad \lim_{n \rightarrow \infty} \sup_{|x| \leq r} \left| \int_{|y| < (l+1)r} \frac{\rho_{\text{Ai},\beta}^{n,1}(y) - \rho_{\text{Ai},\beta}^1(y) - (\hat{\varrho}^n(y) - \hat{\varrho}(y))}{x - y} dy \right| = 0.$$

From Lemma 4.3 and the fact that  $\rho_{\text{Ai},\beta}^{n,1}(x + 2n^{2/3})$  and  $\hat{\varrho}^n(x + 2n^{2/3})$  are symmetric



functions of  $x$ , we deduce that for each  $l \in \mathbb{N}$

$$\begin{aligned}
(4.14) \quad & \lim_{n \rightarrow \infty} \limsup_{s \rightarrow \infty} \sup_{|x| \leq r} \left| \int_{(l+1)r \leq |y| < s} \frac{\rho_{\text{Ai},\beta}^{n,1}(y) - \rho_{\text{Ai},\beta}^1(y) - (\hat{\rho}^n(y) - \hat{\rho}(y))}{x - y} dy \right| \\
& \leq \lim_{n \rightarrow \infty} \limsup_{s \rightarrow \infty} \sup_{|x| \leq r} \int_{(l+1)r \leq |y| < s} \frac{c_3(1 + \mathbf{1}(-2n^{2/3} \leq y))}{|y|^{1/4}|x - y|} dy \\
& \quad + \lim_{n \rightarrow \infty} \limsup_{s \rightarrow \infty} \sup_{|x| \leq r} \int_{-s}^{-2n^{2/3}} \frac{c_3}{(|y + 4n^{2/3}| \vee 1)^{1/4}|x - y|} dy \\
& = \int_{(l+1)r \leq |y| < \infty} \frac{2c_3}{|y|^{1/4}(|y| - r)} dy = \mathcal{O}(l^{-1/4}).
\end{aligned}$$

Putting (4.12), (4.13) and (4.14) together we obtain (4.10).

Eq. (4.11) follows directly from a straightforward calculation.  $\square$

We use Lemma 4.3 and Lemma 4.4 in the proof of Theorem 2.1 in Subsection 5.1.

Next, we turn to the preparation of the proof of Theorem 2.2 in Subsection 5.2. Proposition 4.6 below is a primary step in the proof of Theorem 2.2. To prove Proposition 4.6 we present Lemma 4.5. Since the proof of Lemma 4.5 is quite long, it is given in Subsection 6.5.

Let  $[q]^{[0]}$  denote the complex scalar part of the quaternion  $q$  defined as in Subsection 6.1. We also set  $[q]^{[0]} = q$  if  $q$  is a complex number. Let  $K_{\text{Ai},\beta}^n$  be as in (6.21) and (6.22). We remark that  $K_{\text{Ai},\beta}^n(x, x)$  are scalar quaternions, and can be regarded as positive numbers. Hence we have  $\rho_{\text{Ai},\beta}^{n,1}(x) = K_{\text{Ai},\beta}^n(x, x)^{[0]} = K_{\text{Ai},\beta}^n(x, x)$ . To simplify the notation we set

$$(4.15) \quad L_{\beta}^n(x, y) = K_{\text{Ai},\beta}^n(y, x) K_{\text{Ai},\beta}^n(x, y).$$

In Lemma 6.7 we prove that  $L_{\beta}^n(x, y)$  is also a scalar quaternion. Hence we set  $L_{\beta}^n(x, y) = L_{\beta}^n(x, y)^{[0]}$ , which simplifies the representation of  $I_{\beta,2}^n$ ,  $I_{\beta,3}^n$  and  $I_{\beta,4}^n$  although this is not essential in our proof.

**Lemma 4.5.** Assume  $\beta = 1, 2, 4$ . Set

$$\begin{aligned}
I_{\beta,1}^n(x, s) &= \int_{|x-u|>s} \frac{\rho_{\text{Ai},\beta}^{n,1}(u)}{|x-u|^2} du, & I_{\beta,2}^n(x, s) &= \int_{|x-u|>s} \frac{|L_\beta^n(x, u)|}{|x-u|^2} du, \\
I_{\beta,3}^n(x, s) &= \int_{|x-u|>s} \frac{|L_\beta^n(x, u)|}{|x-u|} du, & I_{\beta,4}^n(x, s) &= \int_{\substack{|x-u|>s \\ |x-v|>s}} \frac{|L_\beta^n(u, v)|}{|x-u||x-v|} dudv, \\
I_{\beta,5}^n(x, s) &= \int_{\substack{|x-u|>s \\ |x-v|>s}} \frac{|[K_{\text{Ai},\beta}^n(u, x)K_{\text{Ai},\beta}^n(x, v)K_{\text{Ai},\beta}^n(v, u)]^{[0]}|}{|x-u||x-v|} dudv, \\
I_{\beta,6}^n(x, s) &= \int_{\substack{|x-u|>s \\ |x-v|>s}} \frac{|[K_{\text{Ai},\beta}^n(u, v)K_{\text{Ai},\beta}^n(v, x)K_{\text{Ai},\beta}^n(x, u)]^{[0]}|}{|x-u||x-v|} dudv.
\end{aligned}$$

Then

$$(4.16) \quad \lim_{s \rightarrow \infty} \sup_{2 \leq n \in \mathbb{N}} \sup_{|x| \leq r} I_{\beta,k}^n(x, s) = 0 \quad \text{for all } 1 \leq k \leq 6.$$

We now state Proposition 4.6.

**Proposition 4.6** (Key estimate). Assume  $\beta = 1, 2, 4$ . We set  $\mathbf{y} = \sum_i \delta_{y_i}$  and

$$(4.17) \quad w_{\beta,s}^n(x, \mathbf{y}) = \sum_{|x-y_i| \geq s} \frac{1}{x-y_i} - \int_{|x-y| \geq s} \frac{\rho_{\text{Ai},\beta,x}^{n,1}(y)}{x-y} dy.$$

Then,

$$(4.18) \quad \lim_{s \rightarrow \infty} \sup_{2 \leq n \in \mathbb{N}} \sup_{|x| \leq r} E^{\mu_{\text{Ai},\beta,x}^n} [|w_{\beta,s}^n(x, \cdot)|^2] = 0 \quad \text{for each } r \in \mathbb{N}.$$

*Proof.* First, we note that  $E^{\mu_{\text{Ai},\beta,x}^n} [w_{\beta,s}^n(x, \cdot)] = 0$ . Hence,

$$(4.19) \quad E^{\mu_{\text{Ai},\beta,x}^n} [|w_{\beta,s}^n(x, \cdot)|^2] = \text{Var}^{\mu_{\text{Ai},\beta,x}^n} [w_{\beta,s}^n(x, \cdot)].$$

From the standard calculation of correlation functions and determinantal kernels, and the fact that  $K_{\text{Ai},\beta,x}^n(u, v)K_{\text{Ai},\beta,x}^n(v, u)$  is a scalar quaternion, we deduce that

$$\begin{aligned}
(4.20) \quad \text{Var}^{\mu_{\text{Ai},\beta,x}^n} [w_{\beta,s}^n(x, \cdot)] &= \int_{|x-u|>s} \frac{\rho_{\text{Ai},\beta,x}^{n,1}(u)}{(x-u)^2} du \\
&\quad - \int_{\substack{|x-u|>s \\ |x-v|>s}} \frac{K_{\text{Ai},\beta,x}^n(u, v)K_{\text{Ai},\beta,x}^n(v, u)}{(x-u)(x-v)} dudv.
\end{aligned}$$

From relation (4.2) with a direct calculation we see that

$$(4.21) \quad \rho_{\text{Ai},\beta,x}^{n,1}(u) = \rho_{\text{Ai},\beta}^{n,1}(u) - \frac{L_{\beta}^n(x, u)}{\rho_{\text{Ai},\beta}^{n,1}(x)}$$

and that

$$(4.22) \quad \begin{aligned} K_{\text{Ai},\beta,x}^n(u, v) K_{\text{Ai},\beta,x}^n(v, u) &= L_{\beta}^n(v, u) + \frac{L_{\beta}^n(x, u) L_{\beta}^n(v, x)}{\rho_{\text{Ai},\beta}^{n,1}(x)^2} \\ &\quad - \frac{K_{\text{Ai},\beta}^n(u, v) K_{\text{Ai},\beta}^n(v, x) K_{\text{Ai},\beta}^n(x, u)}{\rho_{\text{Ai},\beta}^{n,1}(x)} \\ &\quad - \frac{K_{\text{Ai},\beta}^n(u, x) K_{\text{Ai},\beta}^n(x, v) K_{\text{Ai},\beta}^n(v, u)}{\rho_{\text{Ai},\beta}^{n,1}(x)}. \end{aligned}$$

Combining (4.20), (4.21) and (4.22) with (4.3) and  $I_{\beta,i}^n$  in Lemma 4.5 we obtain that

$$(4.23) \quad \begin{aligned} \text{Var}^{\mu_{\text{Ai},\beta,x}^n}[w_{\beta,s}^n(x, \cdot)] &\leq I_{\beta,1}^n(x, s) + I_{\beta,4}^n(x, s) \\ &\quad + \frac{I_{\beta,2}^n(x, s)}{\rho_{\text{Ai},\beta}^{n,1}(x)} + \left( \frac{I_{\beta,3}^n(x, s)}{\rho_{\text{Ai},\beta}^{n,1}(x)} \right)^2 + \frac{I_{\beta,5}^n(x, s)}{\rho_{\text{Ai},\beta}^{n,1}(x)} + \frac{I_{\beta,6}^n(x, s)}{\rho_{\text{Ai},\beta}^{n,1}(x)}. \end{aligned}$$

Recall that  $\rho_{\text{Ai},\beta}^{n,1}$  are positive continuous functions, and converge to  $\rho_{\text{Ai},\beta}^1$  compact uniformly in  $x$  as  $n \rightarrow \infty$  by Lemma 4.2. Note that  $\rho_{\text{Ai},\beta}^1$  is locally uniformly positive. Then we deduce that

$$(4.24) \quad \inf_{2 \leq n \in \mathbb{N}} \inf_{|x| \leq r} \rho_{\text{Ai},\beta}^{n,1}(x) > 0 \quad \text{for each } r \in \mathbb{N}.$$

We have therefore, deduced (4.18) from (4.19), (4.23), (4.24), and Lemma 4.5.  $\square$

## 5 Proof of main theorems

### 5.1 Proof of Theorem 2.1: Unlabeled diffusions related to Airy RPFs

In this section, we prove Theorem 2.1. We begin with the quasi-Gibbs property (A.3.2). For this we quote a result from [29, 30]. In the following, we take  $d = 1$  and  $S = \mathbb{R}$ . Thus  $\mathbf{S}$  is the configuration space over  $\mathbb{R}$ . In addition to (A.3.1), we introduce three further conditions (A.5.1)–(A.5.3) for the quasi-Gibbs property. These conditions guarantee that  $\mu$  has a good finite-particle approximation  $\{\mu^n\}_{n \in \mathbb{N}}$ , which enables us to prove the quasi-Gibbs property of  $\mu$ .

(A.5.1) There exists a sequence of RPFs  $\{\mu^n\}_{n \in \mathbb{N}}$  on  $S$  satisfying the following.

(1) The  $n$ -correlation functions  $\rho^{n,n}$  of  $\mu^n$  satisfy

$$(5.1) \quad \lim_{n \rightarrow \infty} \rho^{n,n}(\mathbf{x}_n) = \rho^n(\mathbf{x}_n) \quad \text{a.e.} \quad \text{for all } n \in \mathbb{N},$$

$$(5.2) \quad \sup\{\rho^{n,n}(\mathbf{x}_n); n \in \mathbb{N}, \mathbf{x}_n \in \{|x| < r\}^n\} \leq \{c_4 n^\delta\}^n \quad \text{for all } n, r \in \mathbb{N},$$

where  $c_4 = c_4(r) > 0$  and  $\delta = \delta(r) < 1$  are constants depending on  $r \in \mathbb{N}$ .

(2)  $\mu^n(s(\mathbb{R}) \leq n_n) = 1$  for some  $n_n \in \mathbb{N}$ .

(3)  $\mu^n$  is a  $(\Phi^n, -\beta \log|x - y|)$ -canonical Gibbs measure, where  $0 < \beta < \infty$ .

(A.5.2) There exists a sequence  $\{\mathbf{m}_\infty^n\}_{n \in \mathbb{N}}$  in  $\mathbb{R}$  such that

$$(5.3) \quad \lim_{n \rightarrow \infty} \{\Phi^n(x) - \mathbf{m}_\infty^n x\} = \Phi(x) \quad \text{for a.e. } x,$$

$$(5.4) \quad \inf_{n \in \mathbb{N}} \inf_{|x| < r} \{\Phi^n(x) - \mathbf{m}_\infty^n x\} > -\infty \quad \text{for each } r \in \mathbb{N}.$$

(A.5.3) There exists a sequence  $\{\mathbf{m}_r^n\}_{n,r \in \mathbb{N}}$  in  $\mathbb{R}$  such that

$$(5.5) \quad \lim_{r \rightarrow \infty} \mathbf{m}_r^n = \mathbf{m}_\infty^n \quad \text{for all } n \in \mathbb{N},$$

$$(5.6) \quad \sup_{n \in \mathbb{N}} |\mathbf{m}_r^n| < \infty \quad \text{for all } r \in \mathbb{N},$$

$$(5.7) \quad \lim_{r \rightarrow \infty} \sup_{n \in \mathbb{N}} \left\| \left\{ \beta \sum_{r \leq |s_i| < \infty} \frac{1}{s_i} \right\} + (\mathbf{m}_\infty^n - \mathbf{m}_r^n) \right\|_{L^2(S, \mu^n)} = 0 \quad (s = \sum_i \delta_{s_i}).$$

Moreover, the sequence of one-correlation functions  $\rho^{n,1}$  of  $\mu^n$  satisfies

$$(5.8) \quad \sup_{n \in \mathbb{N}} \left\{ \int_{1 \leq |x|} \frac{1}{|x|^2} \rho^{n,1}(x) dx \right\} < \infty.$$

The next lemma has been proved in [30, Theorem 2.2].

**Lemma 5.1.** *Let  $\beta \in (0, \infty)$ . Assume (A.3.1) and (A.5.1)–(A.5.3). Then,  $\mu$  is a quasi-Gibbsian with potential  $(\Phi, -\beta \log|x - y|)$ .*

We now apply Lemma 5.1 to Airy RPFs. We achieve this through a sequence of lemmas, in which we assume  $\beta = 1, 2$  or  $4$ .

**Lemma 5.2.** *Let  $\mu_{\text{Ai},\beta}^n$  be the RPF on  $\mathbb{R}$  whose labeled distribution is given by  $m_{\text{Ai},\beta}^n$  in (1.3). Set  $n_n = n$  and  $\Phi^n(x) = \frac{\beta}{4} n^{-1/3} x^2 + \beta n^{1/3} x$ . Then, (A.5.1) holds.*

*Proof.* (1) of (A.5.1) is well known (see [6], for example). Moreover, (2) and (3) of (A.5.1) are obvious from (1.3).  $\square$

**Lemma 5.3.** (1) *The following limit exists compact uniformly in  $C(\mathbb{R})$ .*

$$(5.9) \quad u_\beta(x) = \lim_{s \rightarrow \infty} \left\{ \int_{|y| < s} \frac{\rho_{\text{Ai}, \beta, x}^1(y)}{x - y} dy - \int_{|y| < s} \frac{\hat{\varrho}(y)}{-y} dy \right\}.$$

(2) *Let  $u_\beta^n$  ( $n \in \mathbb{N}$ ) be the continuous functions defined as*

$$(5.10) \quad u_\beta^n(x) = \int_{\mathbb{R}} \frac{\rho_{\text{Ai}, \beta, x}^{n,1}(y)}{x - y} dy - n^{1/3} - \frac{n^{-1/3}}{2}x.$$

*Then,  $u_\beta^n$  converges to  $u_\beta$  compact uniformly in  $C(\mathbb{R})$ .*

*Proof.* Eq. (5.9) follows directly from (4.7) and (6.14). Indeed,

$$\begin{aligned} & \sup_{s \leq t < \infty} \left| \int_{s \leq |y| < t} \frac{\rho_{\text{Ai}, \beta, x}^1(y)}{x - y} dy - \int_{s \leq |y| < t} \frac{\hat{\varrho}(y)}{-y} dy \right| \\ & \leq \sup_{s \leq t < \infty} \int_{s \leq |y| < t} \left| \frac{\rho_{\text{Ai}, \beta, x}^1(y)}{x - y} - \frac{\hat{\varrho}(y)}{x - y} \right| + \left| \frac{\hat{\varrho}(y)}{x - y} - \frac{\hat{\varrho}(y)}{-y} \right| dy \\ & \leq \int_{s \leq |y| < \infty} \frac{c_3 + c_5}{|x - y||y|^{1/4}} + \frac{|\hat{\varrho}(y)|x|}{|x - y||y|} dy = \mathcal{O}(s^{-1/4}) \quad (s \rightarrow \infty). \end{aligned}$$

Here we used the definition  $\hat{\varrho}(y) = \frac{1_{(-\infty, 0]}\sqrt{-y}}{\pi}$  in the last line.

Recall that  $n^{1/3} = - \int_{\mathbb{R}} (\hat{\varrho}^n(y)/y) dy$  by (1.15). Then (5.9) and (5.10) yields that

$$\begin{aligned} & |u_\beta^n(x) - u_\beta(x)| \\ & = \left| \int_{\mathbb{R}} \frac{\rho_{\text{Ai}, \beta, x}^{n,1}(y)}{x - y} dy - n^{1/3} - \frac{n^{-1/3}}{2}x \right. \\ & \quad \left. - \lim_{s \rightarrow \infty} \left\{ \int_{|y| < s} \frac{\rho_{\text{Ai}, \beta, x}^1(y)}{x - y} dy - \int_{|y| < s} \frac{\hat{\varrho}(y)}{-y} dy \right\} \right| \\ & = \left| \lim_{s \rightarrow \infty} \left\{ \int_{|y| < s} \frac{\rho_{\text{Ai}, \beta, x}^{n,1}(y) - \rho_{\text{Ai}, \beta, x}^1(y)}{x - y} - \frac{\hat{\varrho}^n(y) - \hat{\varrho}(y)}{-y} dy \right\} - \frac{n^{-1/3}}{2}x \right| \\ & \leq \lim_{s \rightarrow \infty} \left| \int_{|y| < s} \frac{\rho_{\text{Ai}, \beta, x}^{n,1}(y) - \rho_{\text{Ai}, \beta, x}^1(y) - (\hat{\varrho}^n(y) - \hat{\varrho}(y))}{x - y} dy \right| \\ & \quad + \lim_{s \rightarrow \infty} \left| \int_{|y| < s} (\hat{\varrho}^n(y) - \hat{\varrho}(y)) \left\{ \frac{1}{x - y} - \frac{1}{-y} \right\} dy \right| + \frac{n^{-1/3}}{2}|x|. \end{aligned}$$

Hence, applying (4.10) and (4.11) to the last two lines, we obtain (2).  $\square$

For  $r \in \mathbb{N} \cup \{\infty\}$ , we set

$$(5.11) \quad \mathfrak{m}_r^n = \beta \int_{|y| < r} \frac{\rho_{\text{Ai}, \beta, 0}^{n,1}(y)}{-y} dy.$$

**Lemma 5.4.** *Let  $\Phi^n$  and  $u_\beta$  be as in Lemma 5.2 and Lemma 5.3, respectively. Let  $\mathfrak{m}_\infty^n$  be as in (5.11). Let  $\Phi(x) = -\beta u_\beta(0)x$ . Then, (A.5.2) holds.*

*Proof.* Let  $u_\beta^n$  be the continuous function defined as (5.10). Then, from Lemma 5.3 and (5.11), we deduce that

$$(5.12) \quad \lim_{n \rightarrow \infty} \{\beta n^{1/3} - \mathfrak{m}_\infty^n\} = \lim_{n \rightarrow \infty} -\beta u_\beta^n(0) = -\beta u_\beta(0).$$

Recall that  $\Phi^n(x) = \frac{\beta}{4} n^{-1/3} x^2 + \beta n^{1/3} x$  by definition. Then deduce from (5.12) that

$$(5.13) \quad \begin{aligned} \lim_{n \rightarrow \infty} \{\Phi^n(x) - \mathfrak{m}_\infty^n x\} &= \lim_{n \rightarrow \infty} \left\{ \left( \frac{\beta}{4} n^{-1/3} x^2 + \beta n^{1/3} x \right) - \mathfrak{m}_\infty^n x \right\} \\ &= -\beta u_\beta(0)x. \end{aligned}$$

Hence, (A.5.2) follows directly from (5.12) and (5.13).  $\square$

**Lemma 5.5.** *Let  $\mathfrak{m}_r^n$  and  $\mathfrak{m}_\infty^n$  be as in (5.11). Then, (A.5.3) holds with  $\rho^{n,1} = \rho_{\text{Ai}, \beta}^{n,1}$ .*

*Proof.* Recall that  $\int_{\mathbb{R}} \rho_{\text{Ai}, \beta, 0}^{n,1}(y) dy = n - 1$ . Hence, (5.5) follows from

$$|\mathfrak{m}_\infty^n - \mathfrak{m}_r^n| = \left| \beta \int_{r \leq |y|} \frac{\rho_{\text{Ai}, \beta, 0}^{n,1}(y)}{-y} dy \right| \leq \frac{\beta(n-1)}{r} \rightarrow 0 \quad \text{as } r \rightarrow \infty.$$

Moreover, we deduce (5.6) from the compact uniform convergence of  $\rho_{\text{Ai}, \beta, 0}^{n,1}$  and  $\{\rho_{\text{Ai}, \beta, 0}^{n,1}\}'$  to  $\rho_{\text{Ai}, \beta, 0}^1$  and  $\{\rho_{\text{Ai}, \beta, 0}^1\}'$ , respectively. We deduce (5.7) from Proposition 4.6 and (5.11) easily. Finally, from (4.6) we deduce that

$$\begin{aligned} \int_{1 \leq |x|} \frac{\rho_{\text{Ai}, \beta}^{n,1}(x)}{|x|^2} dx &\leq \int_{1 \leq |x|} \left\{ \frac{|\rho_{\text{Ai}, \beta}^{n,1}(x) - \hat{\varrho}^n(x)|}{|x|^2} + \frac{\hat{\varrho}^n(x)}{|x|^2} \right\} dx \\ &\leq \int_{1 \leq |x|} \left\{ \frac{c_3}{|x|^2} + \frac{\sqrt{|x|}}{\pi |x|^2} \right\} dx. \end{aligned}$$

This implies (5.8) directly.  $\square$

**Theorem 5.6.** *Let  $\beta = 1, 2, 4$  and  $u_\beta$  be in (5.9). Set*

$$\Phi_\beta(x) = \beta \int_x^0 u_\beta(y) dy, \quad \Psi_\beta(x, y) = -\beta \log |x - y|.$$

*Then, Airy RPF  $\mu_{\text{Ai}, \beta}$  is a  $(\Phi_\beta, \Psi_\beta)$ -quasi Gibbs measure.*

*Proof.* Let  $\Phi = -\beta u_\beta(0)x$ . Since  $F(x) = \Phi_\beta(x) - \Phi(x)$  is a locally bounded measurable function by  $u_\beta \in C(\mathbb{R})$ , Theorem 5.6 is equivalent to showing that  $\mu_{\text{Ai},\beta}$  is a  $(\Phi, \Psi_\beta)$ -quasi Gibbs measure in Remark 3.1 (1). We first note that (A.3.1) is obvious from Lemma 4.1. We deduce (A.5.1)–(A.5.3) from Lemma 5.2, Lemma 5.4, and Lemma 5.5, respectively. Hence, the claim follows from Lemma 5.1.  $\square$

*Proof of Theorem 2.1.* By Lemma 4.1 we see that  $\mu_{\text{Ai},\beta}^n$  satisfies (A.3.1). The quasi-Gibbs property (A.3.2) follows from Theorem 5.6. Hence, the assumptions of Lemma 3.1 are fulfilled. In particular, (1) and (2) of Theorem 2.1 follow directly from Lemma 3.1.

Next, we prove (3) of Theorem 2.1. From Lemma 4.1 and Theorem 5.6 we deduce (A.3.3) and (A.3.4). Eq. (2.8) follows from (A.3.4). Eq. (2.7) is obvious because we take  $\mathcal{S}_{\mu_{\text{Ai},\beta}}$  as state space of the diffusion. To take  $\mathcal{S}_{\mu_{\text{Ai},\beta}}$  as  $\mathcal{S}_{\mu_{\text{Ai},\beta}} \subset \mathcal{S}_{\text{s.i.}}^{+f}$ , we note the identity

$$(5.14) \quad X_t(\mathbb{R}^+) = X_t([0, \max\{X_t^1, 0\}]).$$

In fact, this is obvious because  $X_t^1$  is the position of the top particle at time  $t$ .

From (5.14) and (2.8) we see that

$$P_s(X_t(\mathbb{R}^+) < \infty \text{ for all } t) = 1 \quad \text{for all } s \in \mathcal{S}_{\mu_{\text{Ai},\beta}}.$$

From this and (2.3) we obtain (2.6).  $\square$

## 5.2 Proof of Theorem 2.2: ISDEs of Airy interacting Brownian motions

We prove Theorem 2.2 by applying Lemma 3.2 to  $\mu_{\text{Ai},\beta}$ . For this we confirm that  $\mu_{\text{Ai},\beta}$  satisfies the conditions (A.3.1)–(A.3.5) in Section 3. We note that, for  $\mu_{\text{Ai},\beta}$ , we have already proved (A.3.1), (A.3.3), and (A.3.4) in Lemma 4.1 and (A.3.2) in Theorem 5.6.

We begin by proving the existence of the logarithmic derivative  $\mathbf{d}^{\mu_{\text{Ai},\beta}}$  of  $\mu_{\text{Ai},\beta}$  and its explicit representation. Let  $\mu_{\text{Ai},\beta}^{[1]}$  be the 1-Campbell measure of  $\mu_{\text{Ai},\beta}^{[1]}$ . We write  $\mathbf{y} = \sum \delta_{y_j}$  below.

**Theorem 5.7.** *For each  $\beta = 1, 2$ , and 4, the logarithmic derivative  $\mathbf{d}^{\mu_{\text{Ai},\beta}}$  exists in  $L_{\text{loc}}^p(\mu_{\text{Ai},\beta}^{[1]})$  for some  $p > 1$  and is given by*

$$(5.15) \quad \mathbf{d}^{\mu_{\text{Ai},\beta}}(x, \mathbf{y}) = \beta \lim_{s \rightarrow \infty} \left\{ \sum_{|x - y_j| < s} \frac{1}{x - y_j} - \int_{|y| < s} \frac{\hat{\rho}(y)}{-y} dy \right\}.$$

To prove Theorem 5.7 we divide  $\mathbf{d}^{\mu_{\text{Ai},\beta}}$  as follows:

$$(5.16) \quad \mathbf{d}^{\mu_{\text{Ai},\beta}}(x, y) = \beta \{u_\beta(x) + \lim_{s \rightarrow \infty} \mathbf{g}_{\beta,s}(x, y)\}.$$

Here  $u_\beta$  is the continuous function defined by (5.9) and

$$(5.17) \quad \mathbf{g}_{\beta,s}(x, y) = \sum_{|x-y_j| < s} \frac{1}{x-y_j} - \int_{|x-y| < s} \frac{\rho_{\text{Ai},\beta,x}^1(y)}{x-y} dy.$$

We remark that the convergence of  $\mathbf{g}_{\beta,s}$  as  $s \rightarrow \infty$  is not trivial, and is proved in the proof of Theorem 5.7.

We calculate the logarithmic derivatives of the finite particle approximation  $\{\mu_{\text{Ai},\beta}^n\}$ . From (1.3) we easily deduce that the logarithmic derivative  $\mathbf{d}^{\mu_{\text{Ai},\beta}^n}$  of  $\mu_{\text{Ai},\beta}^n$  is

$$(5.18) \quad \mathbf{d}^{\mu_{\text{Ai},\beta}^n}(x, y) = \beta \left\{ \sum_{j=1}^{n-1} \frac{1}{x-y_j} - n^{1/3} - \frac{n^{-1/3}}{2} x \right\}.$$

Here  $y = \sum_{j=1}^{n-1} \delta_{y_j}$  because  $\mu_{\text{Ai},\beta}^n(\{\mathbf{s}(S) = n\}) = 1$ .

Let  $u_\beta^n(x)$  and  $w_{\beta,s}^n(x, y)$  be as in (5.10) and (4.17), respectively. Let

$$\mathbf{g}_{\beta,s}^n(x, y) = \sum_{|x-y_j| < s} \frac{1}{x-y_j} - \int_{|x-y| < s} \frac{\rho_{\text{Ai},\beta,x}^{n,1}(y)}{x-y} dy.$$

Then, from (5.18) we deduce that

$$(5.19) \quad \mathbf{d}^{\mu_{\text{Ai},\beta}^n}(x, y) = \beta \{u_\beta^n(x) + \mathbf{g}_{\beta,s}^n(x, y) + w_{\beta,s}^n(x, y)\}.$$

Furthermore, we have the following:

**Lemma 5.8.** *Assume  $\beta = 1, 2, 4$ . Then, for some  $\hat{p} > 1$*

$$(5.20) \quad \lim_{n \rightarrow \infty} u_\beta^n(x) = u_\beta(x) \text{ in } L_{\text{loc}}^{\hat{p}}(\mathbb{R}, dx),$$

$$(5.21) \quad \lim_{n \rightarrow \infty} \mathbf{g}_{\beta,s}^n(x, y) = \mathbf{g}_{\beta,s}(x, y) \text{ in } L_{\text{loc}}^{\hat{p}}(\mu_{\text{Ai},\beta}^{[1]}) \text{ for any } s > 0,$$

$$(5.22) \quad \lim_{s \rightarrow \infty} \sup_{2 \leq n \in \mathbb{N}} \int_{[-r,r] \times \mathbb{S}} |w_{\beta,s}^n(x, y)|^2 d\mu_{\text{Ai},\beta}^{n,[1]} = 0 \text{ for all } r \in \mathbb{N}.$$

*Proof.* We deduce (5.20) from Lemma 5.3. Eq.(5.21) follows from (4.10) in Lemma 4.4, with (1.14), while (5.22) follows from (4.18) in Proposition 4.6.  $\square$



*Proof of Theorem 5.7.* We use [28, Theorem 45] to prove Theorem 5.7. Indeed, from Lemma 5.8, (4.11), (5.2), and (5.19) we see that the assumptions of [28, Theorem 45] are fulfilled. Hence, we deduce (5.16) from [28, Theorem 45]. Then, we can easily see from (5.16) that the logarithmic derivative has the expression in (5.15).  $\square$

*Proof of Theorem 2.2.* At the beginning of this section, we checked (A.3.1)–(A.3.4). By Theorem 5.7 we see that (A.3.5) is satisfied by the logarithmic derivative given by (5.15). Hence, Theorem 2.2 (1) follows directly from Lemma 3.2. Since  $\mathfrak{l}$  gives the injection from the support of the  $\mu_{\text{Ai},\beta}$  to  $\mathbb{R}_{>}^{\mathbb{N}}$ , we obtain Theorem 2.2 (2). Theorem 2.2 (3) is obvious because the Tracy-Widom distribution is equal to the distribution of the top particle under  $\mu_{\text{Ai},\beta}$ .  $\square$

### 5.3 Proof of Theorem 2.3: Strong solutions and pathwise uniqueness

In this section we prove Theorem 2.3 by applying Lemma 3.3 and Lemma 3.5. For this, it is enough to check that  $\mu_{\text{Ai},\beta}$  satisfies conditions (A.3.1)–(A.3.8) given in Section 3. Conditions (A.3.1)–(A.3.5) have already been checked in Subsection 5.2. (A.3.6) is derived from the following lemma.

**Lemma 5.9.** *Let  $\beta = 1, 2, 4$ . Then,  $\mu_{\text{Ai},\beta}$  satisfies (A.3.6).*

*Proof.* The bound (4.9) yields (A.3.6) directly.  $\square$

It only remains to prove (A.3.7) and (A.3.8) to complete the proof of Theorem 2.3. We deduce these assumptions using the results in [32, Subsection 8.3], where (A.3.7) and (A.3.8) correspond, respectively, with (A6) and (A7) in [32]. From [32, Lemma 8.7] we derive (A.3.7) and (A.3.8) checking assumptions (E1) and (E2) in [32], since other conditions (A2)–(A4) in [32] were derived from (A.3.2)–(A.3.4). (See proof of Lemma 3.3).

First we prove (E1). Let  $\mathbf{a} = \{a_k\}_{k \in \mathbb{N}}$  be a sequence of increasing sequences  $a_k = \{a_k(r)\}_{r \in \mathbb{N}}$  of natural numbers such that

$$a_k(r) = kr^3.$$

Let  $K_{k,r} = \{\mathbf{s}; \mathbf{s}(S_r) \leq a_k(r)\}$  and set

$$K[\mathbf{a}] = \bigcup_{k=1}^{\infty} \bigcap_{r=1}^{\infty} K_{k,r}.$$

Then (E1) is obtained from the following.

**Lemma 5.10.** *Let  $\beta = 1, 2, 4$ . Then, the sequence  $\mathbf{a}$  satisfies*

$$(5.23) \quad \mu_{\text{Ai},\beta}(\mathbf{K}[\mathbf{a}]) = 1.$$

*Proof.* Eq. (5.23) follows from (4.9), Borel-Cantelli's lemma, and Chebycheff's inequality. Indeed, we can easily see that

$$\mathbf{K}[\mathbf{a}] = \bigcup_{k=1}^{\infty} \liminf_{r \rightarrow \infty} \mathbf{K}_{k,r}.$$

Hence, from this and the monotonicity of sets  $\mathbf{K}_{k,r}$  in  $k$ , we deduce that

$$(5.24) \quad \mu_{\text{Ai},\beta}(\mathbf{K}[\mathbf{a}]^c) = \mu_{\text{Ai},\beta}\left(\bigcap_{k=1}^{\infty} \limsup_{r \rightarrow \infty} \mathbf{K}_{k,r}^c\right) = \lim_{k \rightarrow \infty} \mu_{\text{Ai},\beta}\left(\limsup_{r \rightarrow \infty} \mathbf{K}_{k,r}^c\right).$$

From Chebycheff's inequality and (4.9), we deduce that

$$\mu_{\text{Ai},\beta}(\mathbf{K}_{k,r}^c) \leq \frac{1}{a_k(r)} E^{\mu_{\text{Ai},\beta}}[\mathbf{s}(S_r)] = \frac{1}{a_k(r)} \int_{S_r} \rho_{\text{Ai},\beta}^1(x) dx = O(r^{-3/2}).$$

Combining this with (5.24) and applying Borel-Cantelli's lemma, we deduce that  $\mu_{\text{Ai},\beta}(\mathbf{K}[\mathbf{a}]^c) = 0$ , which implies (5.23).  $\square$

We next prove **(E2)**. For this we use [32, Lemma 9.2]. Our task is then to check **(F1)** and **(F2)** in [32, Lemma 9.2] with  $\ell = 1$ . In the present situation, **(F1)** is a condition such that

$$\chi_n \mathbf{d}^{\mu_{\text{Ai},\beta}} \in \mathcal{D}^{\mu_{\text{Ai},\beta}^{[1]}}.$$

Here  $\mathcal{D}^{\mu_{\text{Ai},\beta}^{[1]}}$  is the domain of 1-labeled dynamics of Airy interacting Brownian motions, which is the closure of

$$\mathcal{D}^{\mu_{\text{Ai},\beta}^{[1]}} = \{f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{D}_0; \mathcal{E}^{\mu_{\text{Ai},\beta}^{[1]}}(f, f) < \infty, f \in L^2(\mu_{\text{Ai},\beta}^{[1]})\},$$

where  $\mathcal{E}^{\mu_{\text{Ai},\beta}^{[1]}}$  is a bilinear form on  $\mathbb{R} \times \mathbf{S}$  such that, for  $f, g \in C_0^\infty(\mathbb{R}) \otimes \mathcal{D}_0$ ,

$$\mathcal{E}^{\mu_{\text{Ai},\beta}^{[1]}}(f, g) = \int_{\mathbb{R} \times \mathbf{S}} \frac{1}{2} \nabla_x f \cdot \nabla_x g + \mathbb{D}[f, g] d\mu_{\text{Ai},\beta}^{[1]}.$$

Furthermore,  $\chi_n \in C_0^\infty(\mathbb{R}) \otimes \mathcal{D}_0$  is a cut-off function such that  $0 \leq \chi_n \leq 2$ ,  $\chi_n(x, \mathbf{s}) = 1$  on  $\mathbf{H}_n^{[1]}$ , and  $\chi_n(x, \mathbf{s}) = 0$  on  $(\mathbf{H}_{n+1}^{[1]})^c$ . We then obtain **(F1)** from the following lemma.

**Lemma 5.11.** *Let  $\mathbf{s} = \sum_i \delta_{s_i}$  as before. The derivative in  $x$  of  $\mathbf{d}^{\mu_{\text{Ai},\beta}}$  is then given by*

$$(5.25) \quad \nabla_x \mathbf{d}^{\mu_{\text{Ai},\beta}}(x, \mathbf{s}) = -\beta \sum_i \frac{1}{(x - s_i)^2}.$$

Here the sum in (5.25) converges absolutely in  $L^2(\mathbf{H}_n^{[1]}, \mu_{\text{Ai},\beta}^{[1]})$  for all  $n \in \mathbb{N}$ , where  $\mathbf{H}_n^{[1]} = \{(x, \mathbf{s}) \in \mathbb{R} \times \mathbb{S}; x \in S_r, 2^{-n} \leq |x - s_i|\}$ .

*Proof.* This follows directly from the bound (4.9) and the standard calculation of correlation functions determinantal RPFs. Indeed, we see that

$$\begin{aligned} & \int_{\mathbf{H}_n^{[1]}} \left| \sum_i \frac{1}{(x - s_i)^2} \right|^2 d\mu_{\text{Ai},\beta}^{[1]} \\ &= \int_{S_r} \rho_{\text{Ai},\beta}^1(x) dx \left\{ \int_{\mathbb{S}} \left| \sum_{2^{-n} \leq |x - s_i|} \frac{1}{(x - s_i)^2} \right|^2 d\mu_{\text{Ai},\beta,x} \right\} \\ &= \int_{S_r} \rho_{\text{Ai},\beta}^1(x) dx \left\{ \int_{2^{-n} \leq |x-y|, |x-z|} \frac{\rho_{\text{Ai},\beta,x}^2(y, z)}{(x-y)^2(x-z)^2} dy dz \right. \\ & \quad \left. + \int_{2^{-n} \leq |x-y|} \frac{\rho_{\text{Ai},\beta,x}^1(y)}{(x-y)^4} dy \right\} \\ &= \int_{\{x \in S_r, 2^{-n} \leq |x-y|, |x-z|\}} \frac{\rho_{\text{Ai},\beta}^3(x, y, z)}{(x-y)^2(x-z)^2} dx dy dz \\ & \quad + \int_{\{x \in S_r, 2^{-n} \leq |x-y|\}} \frac{\rho_{\text{Ai},\beta}^2(x, y)}{(x-y)^4} dx dy \\ &< \infty. \end{aligned}$$

From this we see the convergence of the sum in (5.25). The equality in (5.25) is obvious by differentiating both sides of (5.15).  $\square$

We see from Lemma 5.11 that **(F2)** with  $\ell = 1$  is satisfied. Indeed, we take  $\mathbf{g}_1$  and  $\mathbf{h}_1$  in [32, **(F2)**] such that  $\mathbf{g}_1(x, s) = 0$  and  $\mathbf{h}_1(x, s) = -\beta/2(x - s)^2$ . We then easily see that, for  $(x, \mathbf{s}) \in \mathbb{R} \times \mathbb{S}$  such that  $\mathbf{s} = \sum_i \delta_{s_i}$ ,

$$(5.26) \quad \sum_i \mathbf{h}_1(x, s_i) \equiv -\sum_i \frac{\beta}{2(x - s_i)^2} \in L^\infty(\mathbf{H}_n^{[1]}, \mu_{\text{Ai},\beta}^{[1]}) \quad \text{for each } n.$$

From (5.26) and Lemma 5.11 we obtain **(F2)**. We have thus completed the proof of Theorem 2.3.

## 5.4 Proof of Theorem 2.6: Girsanov's formula

In this section we complete the proof of Theorem 2.6. Let  $\mathbf{X}$  be the strong solution of (1.9) in Theorem 2.3. Then, condition (A.3.8) is satisfied by the proof of Theorem 2.3 in Subsection 5.3. Hence, for  $\mathbf{P}_s$ -a.s. fixed  $\mathbf{X}^{m*} = \{(X_t^{m+1}, X_t^{m+2}, \dots)\}_{t \in [0, T]}$ , the first  $m$ -components  $\mathbf{X}^m = \{(X_t^1, \dots, X_t^m)\}_{t \in [0, T]}$  become the unique strong solution of the (finite-dimensional) SDE (3.13) with  $\mu = \mu_{Ai, \beta}$  on the interval  $[0, T]$ . That is, under  $\mathbf{P}_s(\mathbf{X}^m \in \cdot | \mathbf{X}_T^{m*})$ ,  $\mathbf{X}^m$  satisfies

$$(5.27) \quad dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \left( \sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|x| < r} \frac{\hat{\rho}(x)}{-x} dx \right\} dt,$$

$$\mathbf{X}_0^m = (s_1, \dots, s_m) \in S^m.$$

We emphasize that  $\mathbf{X}^{m*}$  is regarded as part of the coefficients. Then (2.10) follows from SDE (5.27) and the standard argument of Girsanov's formula.

The second statement (2.11) follows from (A.3.7). Indeed, let  $\mathbf{H}^{[1]}$  be the set as in (3.12). Then, by the definition we see that, when  $\mu = \mu_{Ai, \beta}$ ,

$$\mathbf{H}^{[1]} \subset \bigcup_{k=1}^{\infty} \left\{ (x, s); \left| \lim_{r \rightarrow \infty} \left\{ \left( \sum_{|s_j| < r} \frac{1}{x - s_j} \right) - \int_{|y| < r} \frac{\hat{\rho}(y)}{-y} dy \right\} \right| < k \right\}.$$

Since  $\text{Cap}^{\mu^{[1]}}((\mathbf{H}^{[1]})^c) = 0$  by (3.12), we immediately obtain (2.11).  $\square$

## 6 Appendices

### 6.1 Appendix 1: Quaternion determinant and kernels

We recall the standard quaternion notation for  $2 \times 2$  matrices (see [20, Ch. 2.4]),

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

A quaternion  $q$  is represented by  $q = q^{(0)}\mathbf{1} + q^{(1)}\mathbf{e}_1 + q^{(2)}\mathbf{e}_2 + q^{(3)}\mathbf{e}_3$ , where the  $q^{(i)}$  are complex numbers. There is a natural identification between the  $2 \times 2$  complex matrices and the quaternions given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}(a+d)\mathbf{1} - \frac{i}{2}(a-d)\mathbf{e}_1 + \frac{1}{2}(b-c)\mathbf{e}_2 - \frac{i}{2}(b+c)\mathbf{e}_3.$$

For a quaternion  $q = q^{(0)}\mathbf{1} + q^{(1)}\mathbf{e}_1 + q^{(2)}\mathbf{e}_2 + q^{(3)}\mathbf{e}_3$ , we call  $q^{(0)}$  the complex scalar part of  $q$ . A quaternion is called complex scalar if  $q^{(i)} = 0$  for  $i = 1, 2, 3$ . We often identify a complex scalar quaternion  $q = q^{(0)}\mathbf{1}$  by the complex number  $q^{(0)}$ .

Let  $\bar{q} = q^{(0)}\mathbf{1} - \{q^{(1)}\mathbf{e}_1 + q^{(2)}\mathbf{e}_2 + q^{(3)}\mathbf{e}_3\}$ . A quaternion matrix  $A = [a_{ij}]$  is called a self-dual if  $a_{ij} = \bar{a}_{ji}$  for all  $i, j$ . For a self-dual  $n \times n$  quaternion matrix  $A = [a_{ij}]$ , we set

$$(6.1) \quad \text{qdet} A = \sum_{\sigma \in \mathfrak{S}_n} \text{sign}[\sigma] \prod_{i=1}^{L(\sigma)} [a_{\sigma_i(1)\sigma_i(2)} a_{\sigma_i(2)\sigma_i(3)} \cdots a_{\sigma_i(\ell)\sigma_i(1)}]^{(0)}.$$

Here  $\sigma = \sigma_1 \cdots \sigma_{L(\sigma)}$  denotes a decomposition of  $\sigma$  to products of the cyclic permutations  $\{\sigma_i\}$  with disjoint indices. We write  $\sigma_i = (\sigma_i(1), \dots, \sigma_i(\ell))$ , where  $\ell$  is the length of the cyclic permutation  $\sigma_i$ . The decomposition is unique up to the order of  $\{\sigma_i\}$ . It is known that the right-hand side is well defined (see [20, Section 5.1]).

We now introduce quaternion kernels  $K_{\text{Ai},1}$  and  $K_{\text{Ai},4}$  using the  $2 \times 2$  matrix representation of quaternions.

Let  $K_{\text{Ai},2}$  and  $\text{Ai}$  be as in (1.5) and (1.6), respectively. Let

$$(6.2) \quad \begin{aligned} J_1(x, y) &= K_{\text{Ai},2}(x, y) + \frac{1}{2} \text{Ai}(x) \left(1 - \int_y^\infty \text{Ai}(u) du\right), \\ J_4(x, y) &= K_{\text{Ai},2}(x, y) - \frac{1}{2} \text{Ai}(x) \int_y^\infty \text{Ai}(u) du. \end{aligned}$$

Then, we define quaternion kernels  $K_{\text{Ai},1}$  and  $K_{\text{Ai},4}$  as

$$(6.3) \quad K_{\text{Ai},1}(x, y) = \begin{bmatrix} J_1(x, y) & -\frac{\partial}{\partial y} J_1(x, y) \\ \int_y^x J_1(u, y) du - \frac{1}{2} \text{sign}(x - y) & J_1(y, x) \end{bmatrix},$$

$$(6.4) \quad \frac{1}{2^{\frac{2}{3}}} K_{\text{Ai},4}\left(\frac{x}{2^{\frac{2}{3}}}, \frac{y}{2^{\frac{2}{3}}}\right) = \frac{1}{2} \begin{bmatrix} J_4(x, y) & -\frac{\partial}{\partial y} J_4(x, y) \\ \int_y^x J_4(u, y) du & J_4(y, x) \end{bmatrix}.$$

## 6.2 Appendix 2: Estimates of Airy functions

In this subsection, we collect estimates of Airy function  $\text{Ai}(x)$  and related quantities. We first recall that Airy function satisfies the differential equation

$$(6.5) \quad \text{Ai}''(x) + x \text{Ai}(x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

We use the following asymptotic expansions of Airy functions in the classical sense of Poincaré [41, 23, 24]:

**Lemma 6.1.** *We see that as  $x \rightarrow \infty$*

$$\begin{aligned}\text{Ai}(x) &= \frac{e^{-\frac{2}{3}x^{3/2}}}{2\pi^{1/2}x^{1/4}} \left(1 + \mathcal{O}(x^{-3/2})\right), \\ \text{Ai}'(x) &= -\frac{x^{1/4}e^{-\frac{2}{3}x^{3/2}}}{2\pi^{1/2}} \left(1 + \mathcal{O}(x^{-3/2})\right), \\ \text{Ai}(-x) &= \frac{1}{\pi^{1/2}x^{1/4}} \left[ \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \left(1 + \mathcal{O}(x^{-3/2})\right) \right], \\ \text{Ai}'(-x) &= \frac{x^{1/4}}{\pi^{1/2}} \left[ \sin\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right) \left(1 + \mathcal{O}(x^{-3/2})\right) \right].\end{aligned}$$

**Lemma 6.2.** *As  $x \rightarrow \infty$ , we have the following:*

$$(6.6) \quad \int_0^x \text{Ai}(u)du = \frac{1}{3} - \frac{\exp\{-\frac{2}{3}x^{3/2}\}}{2\sqrt{\pi}x^{3/4}} (1 + o(1)),$$

$$(6.7) \quad \int_{-x}^0 \text{Ai}(u)du = \frac{2}{3} + \mathcal{O}(x^{-3/4}),$$

$$(6.8) \quad 1 - \int_{-x}^\infty \text{Ai}(u)du = \mathcal{O}(x^{-3/4}).$$

We apply the above asymptotic behaviors to examine the one-correlation function  $\rho_{\text{Ai},2}^1$  and the Airy kernels  $K_{\text{Ai},\beta}$  ( $\beta = 1, 2, 4$ ).

**Lemma 6.3.** *As  $x \rightarrow \infty$ , we have the following:*

$$(6.9) \quad \rho_{\text{Ai},2}^1(x) = \mathcal{O}\left(e^{-\frac{4}{3}x^{3/2}}\right),$$

$$(6.10) \quad \rho_{\text{Ai},2}^1(-x) = \frac{\sqrt{x}}{\pi} \{1 + \mathcal{O}(x^{-3/2})\}.$$

*Proof.* From (1.5), (6.5), and the continuity of  $K_{\text{Ai},2}(x, y)$ , we easily deduce that

$$K_{\text{Ai},2}(x, x) = (\text{Ai}'(x))^2 - x(\text{Ai}(x))^2.$$

Combining this with Lemma 6.1 yields Lemma 6.3. □

**Lemma 6.4.** *Let  $x \in \mathbb{R}$  be fixed. Then, it holds that as  $|y| \rightarrow \infty$ ,*

$$(6.11) \quad \left| \int_x^y K_{\text{Ai},2}(u, x)du \right| = \mathcal{O}(1),$$

$$(6.12) \quad K_{\text{Ai},2}(x, y) = \mathcal{O}(|y|^{-3/4}),$$

$$(6.13) \quad \frac{\partial K_{\text{Ai},2}}{\partial y}(x, y) = \mathcal{O}(|y|^{-1/4}).$$

*Proof.* Recall definition (1.5) of  $K_{\text{Ai},2}(x, y)$ . Since

$$\begin{aligned} \int_0^y K_{\text{Ai},2}(u, x) du &= \int_0^y \frac{\text{Ai}(u)\text{Ai}'(x) - \text{Ai}'(u)\text{Ai}(x)}{u - x} du \\ &= \text{Ai}'(x) \int_{-x}^{y-x} \frac{\text{Ai}(x+w)}{w} dw - \text{Ai}(x) \int_{-x}^{y-x} \frac{\text{Ai}'(x+w)}{w} dw, \end{aligned}$$

(6.11) follows from Lemma 6.1. Eq. (6.12) also follows from Lemma 6.1. From (1.5) and (6.5) we deduce that

$$\frac{\partial K_{\text{Ai},2}}{\partial y}(x, y) = \frac{-\text{Ai}'(x)\text{Ai}'(y) + y\text{Ai}(x)\text{Ai}(y)}{x - y} + \frac{K_{\text{Ai},2}(x, y)}{x - y}.$$

Hence (6.13) follows from Lemma 6.1.  $\square$

**Lemma 6.5.** *Let  $\beta = 1, 2, 4$  and  $x \in \mathbb{R}$ . There exists a positive constant  $c_5$  such that, for  $|y| > |x| + 1$ ,*

$$(6.14) \quad |\rho_{\text{Ai},\beta}^1(y) - \rho_{\text{Ai},\beta,x}^1(y)| \leq c_5 \{ |y|^{-3/2} + \mathbf{1}(\beta \neq 2) |y|^{-1/4} \}.$$

Here  $\mathbf{1}(\beta \neq 2) = 1$  for  $\beta = 1, 4$  and  $\mathbf{1}(\beta \neq 2) = 0$  for  $\beta = 2$ .

*Proof.* We set  $L_\beta(x, y) = K_{\text{Ai},\beta}(y, x)K_{\text{Ai},\beta}(x, y)$ . Then by a direct calculation we see that  $L_\beta(x, y)$  is a complex scalar quaternion regarded as a positive number. So we set  $L_\beta(x, y) = [L_\beta(x, y)]^{[0]}$ . From the relation similar to (4.1) and (4.3) we deduce that

$$(6.15) \quad \rho_{\text{Ai},\beta}^1(y) - \rho_{\text{Ai},\beta,x}^1(y) = \frac{L_\beta(y, x)}{\rho_{\text{Ai},\beta}^1(x)}.$$

Since  $\rho_{\text{Ai},\beta}^1$  are continuous and positive, it remains to control  $L_\beta(x, y)$ .

Suppose that  $\beta = 2$ . Then (6.14) is immediate from (6.12) and (6.15).

Suppose that  $\beta = 1$ . Then we see that by (6.3)

$$L_1(x, y) = J_1(x, y)J_1(y, x) - \frac{\partial J_1(x, y)}{\partial y} \left( \int_x^y J_1(u, x) du \right) + \frac{\text{sign}(x - y)}{2}.$$

We deduce from Lemma 6.1, (6.8), (6.11), and (6.12) that

$$\int_x^y J_1(u, x) du = \mathcal{O}(1), \quad J_1(x, y)J_1(y, x) = \mathcal{O}(|y|^{-1}),$$

and from Lemma 6.1 and (6.13) that

$$\frac{\partial}{\partial y} J_1(x, y) = \frac{\partial}{\partial y} K_{\text{Ai},2}(x, y) + \frac{\text{Ai}(x)\text{Ai}(y)}{2} = \mathcal{O}(|y|^{-1/4}).$$

Putting these estimates together into (6.15) we obtain (6.14) for  $\beta = 1$ .

Suppose  $\beta = 4$ . Then we have (6.4)

$$\frac{1}{2^{\frac{4}{3}}} L_4\left(\frac{x}{2^{\frac{2}{3}}}, \frac{y}{2^{\frac{2}{3}}}\right) = J_4(x, y) J_4(y, x) - \frac{\partial J_4(x, y)}{\partial y} \int_x^y J_4(u, x) du.$$

Hence we obtain (6.14) similarly as for the case where  $\beta = 1$ . □

### 6.3 Appendix 3: Determinantal kernels of $n$ -particles

Let  $\widehat{H}_n(x)$  ( $n \in \{0\} \cup \mathbb{N}$ ) be Hermite polynomials;

$$\widehat{H}_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

and  $\phi_n(x)$  be the normalized oscillator function defined as

$$\phi_n(x) = \frac{1}{\sqrt{\sqrt{2\pi}n!}} e^{-x^2/4} \widehat{H}_n(x).$$

Then, for  $n \in \mathbb{N}$ , the distribution  $\mu_{\text{bulk},2}^n$  of  $n$  particles in the GUE system is the determinantal point process with correlation kernel

$$K_{\text{GUE}}^n(x, y) = \sqrt{n} \frac{\phi_n(x) \phi_{n-1}(y) - \phi_{n-1}(x) \phi_n(y)}{x - y}.$$

We set for  $\sigma > 0$  and  $\xi \in \mathbb{R}$

$$\varphi_n^{\sigma, \xi}(x) = \sigma^{1/2} \phi_n\left(\xi + \frac{x}{\sigma}\right).$$

We also introduce the function  $\psi_n$  and  $\psi_n^m$  for  $m \in \mathbb{N}$  defined as

$$(6.16) \quad \psi_n(x) = \psi_n^n(x), \quad \psi_n^m(x) = \varphi_n^{m^{1/6}, 2\sqrt{m}}(x).$$

Then  $\psi_n$  ( $n \in \mathbb{N}$ ) have the following properties (see, for example, p. 101 in [1]):

$$(6.17) \quad \psi_n(x - 2n^{2/3}) \text{ is even (resp. odd) if } n \text{ is even (resp. odd),}$$

$$(6.18) \quad \int_{\mathbb{R}} \psi_{2n-1}(x) dx = 0, \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \psi_{2n}(x) dx = 2,$$

$$(6.19) \quad \psi_n''(x) = \left\{ \frac{x^2}{4n^{2/3}} + x - \frac{1}{2n^{1/3}} \right\} \psi_n(x),$$

and

$$(6.20) \quad n^{1/6} \psi_n'(x) = -\frac{\sqrt{n+1}}{2} \psi_{n+1}^n(x) + \frac{\sqrt{n}}{2} \psi_{n-1}^n(x).$$



Using the function  $\psi_n$ , the correlation kernel of  $\mu_{\text{Ai},2}^n$  is written as

$$\begin{aligned}
(6.21) \quad K_{\text{Ai},2}^n(x, y) &= \frac{1}{n^{1/3}} \sum_{m=1}^{n-1} \psi_m(x) \psi_m(y) \\
&= n^{1/3} \frac{\psi_n(x) \psi_{n-1}^n(y) - \psi_{n-1}^n(x) \psi_n(y)}{x - y} \\
&= \frac{\psi_n(x) \psi_n'(y) - \psi_n'(x) \psi_n(y)}{x - y} - \frac{1}{2n^{1/3}} \psi_n(x) \psi_n(y).
\end{aligned}$$

We also define kernels  $K_{\text{Ai},2}^{n,k}$  similarly as  $K_{\text{Ai},2}^n$  such that

$$\begin{aligned}
K_{\text{Ai},2}^{n,k}(x, y) &= \frac{1}{k^{1/3}} \sum_{m=1}^{n-1} \psi_m^k(x) \psi_m^k(y) \\
&= \frac{n^{1/2}}{k^{1/6}} \frac{\psi_n^k(x) \psi_{n-1}^k(y) - \psi_{n-1}^k(x) \psi_n^k(y)}{x - y} \\
&= \frac{\psi_n^k(x) \{\psi_n^k\}'(y) - \{\psi_n^k\}'(x) \psi_n^k(y)}{x - y} - \frac{1}{2k^{1/3}} \psi_n^k(x) \psi_n^k(y).
\end{aligned}$$

For  $\beta = 1, 4$ , the correlation kernels  $K_{\text{Ai},\beta}^n$  of the determinantal point process  $\mu_{\text{Ai},\beta}^n$  are defined as

$$\begin{aligned}
(6.22) \quad K_{\text{Ai},1}^n(x, y) &= \begin{bmatrix} J_1^n(x, y) & -\frac{\partial}{\partial y} J_1^n(x, y) \\ \int_y^x J_1^n(u, y) du - \frac{1}{2} \text{sign}(x - y) & J_1^n(y, x) \end{bmatrix}, \\
\frac{1}{2^{2/3}} K_{\text{Ai},4}^n\left(\frac{x}{2^{2/3}}, \frac{y}{2^{2/3}}\right) &= \frac{1}{2} \begin{bmatrix} J_4^n(x, y) & -\frac{\partial}{\partial y} J_4^n(x, y) \\ \int_y^x J_4^n(u, y) du & J_4^n(y, x) \end{bmatrix}
\end{aligned}$$

with

$$\begin{aligned}
(6.23) \quad J_1^n(x, y) &= K_{\text{Ai},2}^n(x, y) + \frac{1}{2} \psi_{n-1}^n(x) \varepsilon \psi_n(y) + \frac{\psi_{n-1}^n(x)}{\int_{\mathbb{R}} \psi_{n-1}^n(t) dt} \mathbf{1}(n \text{ is odd}), \\
J_4^n(x, y) &= K_{\text{Ai},2}^{2n+1,2n}(x, y) + \frac{\sqrt{2n+1}}{2(2n)^{1/2}} \psi_{2n}(x) \varepsilon \psi_{2n+1}^{2n}(y).
\end{aligned}$$

(Refer to [1, Section 3.9].) Here, for each integrable real-valued function  $f$  on  $\mathbb{R}$

$$(6.24) \quad (\varepsilon f)(x) = \int_{\mathbb{R}} \frac{1}{2} \text{sign}(x - y) f(y) dy = \frac{1}{2} \int_{\mathbb{R}} f(y) dy - \int_x^\infty f(y) dy.$$

**Lemma 6.6.** *Let  $\beta = 1, 4$ . Then for all  $n \in \mathbb{N}$*

$$(6.25) \quad \frac{\partial}{\partial y} J_\beta^n(x, y) = -\frac{\partial}{\partial x} J_\beta^n(y, x),$$

$$(6.26) \quad \int_y^x J_\beta^n(u, y) du = -\int_x^y J_\beta^n(u, x) du.$$

*Proof.* Let  $\beta = 1$ . Then from (6.23) we see that

$$\begin{aligned}\frac{\partial}{\partial x}J_1^n(y, x) &= \frac{\partial}{\partial x}K_{Ai, 2}^n(y, x) + \frac{1}{2}\psi_{n-1}^n(y)\psi_n(x) \\ &= \frac{\psi_n(y)\psi_n''(x) - \psi_n'(y)\psi_n'(x)}{y - x} - \frac{1}{2n^{1/3}}\psi_n'(x)\psi_n(y) \\ &\quad - \frac{\psi_n(y)\psi_n'(x) - \psi_n'(y)\psi_n(x)}{(y - x)^2} + \frac{1}{2}\psi_{n-1}^n(y)\psi_n(x).\end{aligned}$$

Then we have from (6.19) that

$$\begin{aligned}(6.27) \quad &\frac{\partial}{\partial x}J_1^n(y, x) + \frac{\partial}{\partial y}J_1^n(x, y) \\ &= \frac{\psi_n(y)\psi_n''(x) - \psi_n''(y)\psi_n(x)}{y - x} - \frac{\psi_n'(x)\psi_n(y) + \psi_n(x)\psi_n'(y)}{2n^{1/3}} \\ &\quad + \frac{\psi_{n-1}^n(y)\psi_n(x) + \psi_{n-1}^n(x)\psi_n(y)}{2} \\ &= -\psi_n(x)\psi_n(y)\left\{\frac{x + y}{4n^{2/3} + 1}\right\} - \frac{\psi_n'(x)\psi_n(y) + \psi_n(x)\psi_n'(y)}{2n^{1/3}} \\ &\quad + \frac{\psi_{n-1}^n(y)\psi_n(x) + \psi_{n-1}^n(x)\psi_n(y)}{2}.\end{aligned}$$

The relation  $\phi_{n-1}(x) = n^{-1/2}\{\phi_n'(x) + \frac{x}{2}\phi_n(x)\}$  of Hermite polynomials yields that

$$(6.28) \quad \psi_{n-1}^n(x) = n^{-1/3}\psi_n'(x) + \psi_n(x) + \frac{x}{2n^{2/3}}\psi_n(x)$$

The claim (6.25) for  $\beta = 1$  follows from (6.27) and (6.28). The proof of the case  $\beta = 4$  is similar. We obtain (6.26) from (6.25) and a direct calculation.  $\square$

**Lemma 6.7.** *Let  $L_\beta^n$  be as in (4.15). Let  $\beta = 1, 4$ . Then  $L_\beta^n(x, y)$  are scalar quaternions regarded as real numbers given by*

$$\begin{aligned}(6.29) \quad &L_1^n(x, y) = J_1^n(x, y)J_1^n(y, x) \\ &\quad - \frac{\partial J_1^n(x, y)}{\partial y}\left\{\int_x^y J_1^n(u, x)du - \frac{\text{sign}(y - x)}{2}\right\}, \\ &\frac{1}{2^{4/3}}L_4^n\left(\frac{x}{2^{2/3}}, \frac{y}{2^{2/3}}\right) = J_4^n(x, y)J_4^n(y, x) \\ &\quad - \frac{\partial J_4^n(x, y)}{\partial y}\left\{\int_x^y J_4^n(u, x)du - \frac{\text{sign}(y - x)}{2}\right\}.\end{aligned}$$

*Proof.* Lemma 6.7 follows from Lemma 6.6 immediately.  $\square$

## 6.4 Appendix 4: Estimates of $\psi_n$

In this subsection, we recall the results of Plancherel-Rotach [35], and use them to estimate  $\psi_n$ . The results in this section are used in Subsection 6.5 to complete the proof of Lemma 4.5. The asymptotic behaviors of Hermite polynomials given in [35] are summarized as follows.

**Lemma 6.8.** (i) *If  $x \in (-2n^{2/3}, 0)$ , then for any  $L \in \mathbb{N}$*

$$(6.30) \quad \begin{aligned} \psi_n(x) = & \frac{1 + \mathcal{O}(1/n)}{\pi} \left( \frac{1}{f(x, n)} \right)^{1/4} \\ & \times \sum_{k=0}^{L-1} \sum_{m=0}^k C_{km}(n, \theta) \cos(g(x, n) - c_{km}(\theta)) \\ & + \mathcal{O}(f(x, n)^{-(3L+1)/4}), \end{aligned}$$

where  $\theta = \theta(x, n)$  is the value in  $(0, \pi/2)$  satisfying

$$(6.31) \quad x = 2n^{1/6}(\sqrt{n+1} \cos \theta - \sqrt{n}),$$

and  $f(x, n)$ ,  $g(x, n)$ ,  $c_{km}(\theta)$ , and  $C_{km}(n, \theta)$  are functions defined as

$$(6.32) \quad \begin{aligned} f(x, n) &= n^{2/3} \sin^2 \theta = -x + \frac{x}{n+1} + \frac{n^{2/3}}{n+1} - \frac{x^2 n^{1/3}}{4(n+1)}, \\ g(x, n) &= \frac{n+1}{2}(2\theta - \sin 2\theta), \\ c_{km}(\theta) &= \frac{\theta}{2} - \left(m + \frac{k}{2}\right) \left(\frac{\pi}{2} + \theta\right), \\ C_{km}(n, \theta) &= \frac{1 + (-1)^k}{2} \frac{\Gamma(m + \frac{k+1}{2})}{(n+1)^{k/2} (\sin \theta)^{m+k/2}} a_{km} \end{aligned}$$

with constants  $a_{km}$  independent of  $n$  and  $\theta$ , for instance  $a_{00} = 1$  and  $a_{10} = 0$ .

(ii) *If  $x > 0$ , then for any  $L \in \mathbb{N}$*

$$\begin{aligned} \psi_n(x) = & \frac{1 + \mathcal{O}(n^{-1})}{\pi \sqrt{2(e^{2\theta} - 1)}} \left( \frac{1}{n} \right)^{1/6} \exp \left\{ \left( \frac{n+1}{2} \right) (2\theta - \sinh 2\theta) \right\} \\ & \times \left[ \sum_{k=0}^{L-1} \sum_{m=0}^k \hat{C}_{km}(n, \theta) + \mathcal{O} \left( n^{-L/2} \left( \frac{1 - e^{-2\theta}}{2} \right)^{-3L/2} \right) \right], \end{aligned}$$

where  $\theta = \theta(x, n)$  is the value in  $(0, \infty)$  satisfying

$$x = 2n^{1/6}(\sqrt{n+1} \cosh \theta - \sqrt{n}),$$

and

$$\widehat{C}_{km}(\mathbf{n}, \theta) = \frac{1 + (-1)^k}{2} \frac{\Gamma(m + \frac{k+1}{2})}{(\mathbf{n} + 1)^{k/2}} \left( \frac{2}{e^{-2\theta} - 1} \right)^{m+k/2} a_{km}.$$

(iii) If  $|x| = \mathcal{O}(\mathbf{n}^\varepsilon)$  for some  $\varepsilon \in (0, 1/6)$ , then

$$\psi_{\mathbf{n}}(x) = B\left(x + \mathbf{n}^{-1/3}, \sqrt{2\mathbf{n}} - \frac{x}{\sqrt{2\mathbf{n}^{1/6}}}\right) + \mathcal{O}(\mathbf{n}^{6\varepsilon-1}),$$

where  $B(x, y)$  is a function defined as

$$\begin{aligned} B(x, y) = & \operatorname{Ai}(x) + \left(\frac{2}{y}\right)^{\frac{2}{3}} c_6 x^2 \operatorname{Ai}(x) \\ & + \left(\frac{2}{y}\right)^{\frac{4}{3}} \{c_7 x \operatorname{Ai}(x) + c_8 x^2 \operatorname{Ai}'(x) + c_9 x^4 \operatorname{Ai}(x)\} \end{aligned}$$

with some constants  $c_6, c_7, c_8$ , and  $c_9$  independent of  $x$  and  $y$ .

Using Lemma 6.8 we obtain the following estimate.

**Lemma 6.9.** *There is a positive constant  $c_{10}$  such that*

$$(6.33) \quad |\psi_{\mathbf{n}}(x)| \leq c_{10}(|x| \vee 1)^{-1/4}, \quad x \in [-2\mathbf{n}^{2/3}, \infty), \quad \mathbf{n} \in \mathbb{N},$$

$$(6.34) \quad |\psi'_{\mathbf{n}}(x)| \leq c_{10}(|x| \vee 1)^{1/4}, \quad x \in [-2\mathbf{n}^{2/3}, \infty), \quad \mathbf{n} \in \mathbb{N},$$

$$(6.35) \quad |\psi''_{\mathbf{n}}(x)| \leq c_{10}(|x| \vee 1)^{3/4}, \quad x \in [-2\mathbf{n}^{2/3}, \infty), \quad \mathbf{n} \in \mathbb{N}.$$

*Proof.* We take  $\varepsilon \in (0, 1/9)$  and consider the following three cases:

(I)  $x \in [-2\mathbf{n}^{2/3}, -\mathbf{n}^\varepsilon]$ , (II)  $x \in [-\mathbf{n}^\varepsilon, \mathbf{n}^\varepsilon]$ , and (III)  $x \in [\mathbf{n}^\varepsilon, \infty)$ .

For each case, (6.33) follows from Lemma 6.8, and (6.35) follows from (6.33) with (6.19). Hence it only remains to prove (6.34).

For case (II), let  $L$  be a natural number greater than  $\frac{1}{3}(\frac{4}{3\varepsilon} - 2)$ . From  $\varepsilon \in (0, 1/9)$  we see  $1 - 6\varepsilon > 1/3$ . Then (6.34) is derived from Lemma 6.8 (iii) and Lemma 6.1.

For cases (I) and (III) we use (6.20). In fact, by definition we have

$$\psi_{\mathbf{n} \pm 1}^{\mathbf{n}}(x) = \left(\frac{\mathbf{n}}{\mathbf{n} \pm 1}\right)^{1/12} \psi_{\mathbf{n} \pm 1}\left(\left(\frac{\mathbf{n} \pm 1}{\mathbf{n}}\right)^{1/6} x \pm \frac{2(\mathbf{n} \pm 1)^{1/6}}{\sqrt{\mathbf{n} \pm 1} + \sqrt{\mathbf{n}}}\right).$$

Then this and (6.20) with some calculation deduces (6.34) from

$$(6.36) \quad \mathbf{n}^{1/3} |\psi_{\mathbf{n}+1}(x + \mathbf{n}^{-1/3}) - \psi_{\mathbf{n}-1}(x - \mathbf{n}^{-1/3})| \leq c_{11} |x|^{1/4}, \quad x \in [-2\mathbf{n}^{2/3}, \infty),$$

for some constant  $c_{11}$ . Hence our task is to prove (6.36) for cases (I) and (III).

For case (I), we use Lemma 6.8 (i) with  $L = L(\varepsilon) \in \mathbb{N}$  as above. Recall the relation  $2\sqrt{n} + n^{-1/6}x = 2\sqrt{n+1} \cos \theta$ . Since  $\varepsilon \in (0, 1/9)$ ,

$$(6.37) \quad |f(x, n)|^{-\frac{3L+1}{4}} \leq c_{12}|x|^{1/4-1/(3\varepsilon)} \leq c_{12}n^{-1/3}|x|^{1/4},$$

for some positive constant  $c_{12}$ . Note that for a fixed  $\ell \in \mathbb{Z}$ ,

$$2\sqrt{n} + n^{-1/6}x = \{2\sqrt{n+\ell} + (n+\ell)^{-1/6}(x + \ell n^{-1/3})\}(1 + \mathcal{O}(n^{-1})),$$

and so

$$2\sqrt{n+1} \cos \theta(x, n) = \{2\sqrt{n+\ell+1} \cos \theta(x + \ell n^{-1/3}, n+\ell)\}(1 + \mathcal{O}(n^{-1})).$$

We set  $x_\ell = x + \ell n^{-1/3}$  and  $\theta_\ell = \theta(x + \ell n^{-1/3}, n+\ell)$ . By the same calculation as in the proof of [17, Lemma 5.2], we see that

$$\begin{aligned} |\theta_1 - \theta_{-1}| &= \mathcal{O}((n \sin \theta_0)^{-1}) = \mathcal{O}(n^{-2/3-\varepsilon/2}), \\ |g(x_1, n+1) - g(x_{-1}, n-1)| &= 2\theta_0 + \mathcal{O}((n \sin \theta_0)^{-1}) = \mathcal{O}((\frac{|x|}{n^{2/3}})^{1/2}), \\ |C_{km}(n+\ell, \theta_\ell)| &= \mathcal{O}(|x|^{-3k/4}), \quad \ell = \pm 1, \\ |C_{km}(n+1, \theta_1) - C_{km}(n-1, \theta_{-1})| &= \mathcal{O}(n^{-1/3}|x|^{1/2-3k/4}). \end{aligned}$$

Then, we have

$$\begin{aligned} &\left(\frac{2}{f(x_1, n+1)}\right)^{1/4} \sum_{k=0}^{L-1} \sum_{m=0}^k C_{km}(n+1, \theta_1) \cos(g(x_1, n) - c_{km}(\theta_1)) \\ &- \left(\frac{2}{f(x_{-1}, n-1)}\right)^{1/4} \sum_{k=0}^{L-1} \sum_{m=0}^k C_{km}(n-1, \theta_{-1}) \cos(g(x_{-1}, n-1) - c_{km}(\theta_{-1})) \\ &= \mathcal{O}(n^{-1/3}|x|^{1/4}). \end{aligned}$$

By using the above estimate with (6.37), we have (6.36) from Lemma 6.8 (i).

For case (III), we use Lemma 6.8 (ii), in which the following relation is used:

$$2\sqrt{n} + n^{-1/6}x = 2\sqrt{n+1} \cosh \theta.$$

Note that as  $n \rightarrow \infty$

$$\cosh \theta = \sqrt{\frac{n}{n+1}} + \frac{x}{2n^{1/6}\sqrt{n+1}} = 1 + \frac{x}{2n^{2/3}} + \mathcal{O}(n^{-1})$$

and that as  $x/n^{2/3} \rightarrow 0$

$$\begin{aligned}\theta &= \frac{\sqrt{x}}{n^{1/3}} + \mathcal{O}\left(\frac{x}{n^{2/3}}\right), \\ \sinh \theta &= \frac{\sqrt{x}}{n^{1/3}} + \mathcal{O}\left(\frac{x}{n^{2/3}}\right), \\ 2\theta - \sinh 2\theta &= -\frac{4x^{3/2}}{3n} + \mathcal{O}\left(\left(\frac{x}{n^{2/3}}\right)^2\right).\end{aligned}$$

Then we deduce from Lemma 6.8 (ii) that for  $x = o(n^{2/3})$  with  $n \rightarrow \infty$  and  $x \geq 1$

$$(6.38) \quad \psi_n(x) = \frac{1 + \mathcal{O}(1/n)}{2\sqrt{\pi}} \exp\left(-\left(\frac{2}{3} + o(1)\right)x^{3/2}\right) \quad n \rightarrow \infty.$$

We also deduce from Lemma 6.8 (ii) that for  $1/x = \mathcal{O}(n^{-2/3})$  with  $n \rightarrow \infty$

$$(6.39) \quad |\psi_n(x)| \leq c_{13} \exp\{-c_{14}x^2\}$$

with positive constants  $c_{13}$  and  $c_{14}$ . Hence, we readily obtain (6.36) from (6.38) and (6.39).  $\square$

**Lemma 6.10.** *Let  $\varepsilon\psi_n$  be as in (6.24). Then there exists a constant  $c_{15}$  such that*

$$(6.40) \quad \sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}} |\varepsilon\psi_n(x)| \leq c_{15},$$

$$(6.41) \quad \sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}} \left| \int_a^x \psi_n(y) dy \right| < \infty \quad \text{for all } a \in \mathbb{R}.$$

*Proof.* We deduce (6.40) from (6.18) and (6.41) immediately. Hence our task is to prove (6.41). From (6.33), (6.38) and (6.39) we have

$$(6.42) \quad \sup_{n \in \mathbb{N}} \sup_{-1 \leq x} \left| \int_{-1}^x \psi_n(y) dy \right| < \infty.$$

From (6.42) and the symmetry (6.17) of  $\psi_n$  around  $-2n^{2/3}$ , we deduce (6.41) from

$$(6.43) \quad \sup_{n \in \mathbb{N}} \sup_{-2n^{2/3} \leq x < -1} \left| \int_x^{-1} \psi_n(y) dy \right| < \infty.$$

To prove (6.43) we use Lemma 6.8 (i) with  $L = 2$ . Since  $C_{00} = \sqrt{\pi}$ ,  $C_{10} = 0$ ,  $a_{00} = 1$ , and  $c_{00} = \frac{\theta}{2}$ , we deduce from (6.30) and (6.32) that

$$(6.44) \quad \sup_{n \in \mathbb{N}} \int_{-2n^{2/3}}^{-1} \left| \psi_n(y) - \left(\frac{2}{f(y, n)}\right)^{1/4} \cos(g(y, n) - \frac{\theta}{2}) \right| dy < \infty.$$

Then (6.43) follows from (6.44) and

$$(6.45) \quad \sup_{n \in \mathbb{N}} \sup_{-2n^{2/3} \leq x < -1} \left| \int_x^{-1} \left( \frac{1}{f(y, n)} \right)^{1/4} \cos(g(y, n) - \frac{\theta}{2}) dy \right| < \infty.$$

Hence it only remains to prove (6.45).

Fix  $n \in \mathbb{N}$  and remind that  $\theta \in (0, \pi/2]$ . Then the condition

$$(6.46) \quad 2n^{1/6}(\sqrt{n+1} \cos \theta - \sqrt{n}) \leq -1$$

implies that

$$(6.47) \quad \sin^2 \theta \geq \frac{1 + n^{1/3} - 4^{-1}n^{-1/3}}{n+1}.$$

Taking (6.31) in Lemma 6.8 and (6.46) into account we set

$$\begin{aligned} y &= 2n^{1/6}(\sqrt{n+1} \cos \theta - \sqrt{n}), \\ z &= g(y, n) - \frac{\theta}{2} = \frac{n+1}{2}(2\theta - \sin 2\theta) - \frac{\theta}{2}. \end{aligned}$$

By a straightforward calculation we have

$$(6.48) \quad \frac{\partial y}{\partial \theta} = -2n^{1/6}\sqrt{n+1} \sin \theta,$$

$$(6.49) \quad \frac{\partial z}{\partial \theta} = (n+1)(1 - \cos 2\theta) - 1/2 = 2(n+1) \sin^2 \theta - 1/2,$$

$$(6.50) \quad \frac{\partial y}{\partial \theta} < 0 \text{ and } \frac{\partial z}{\partial \theta} > 0 \text{ for } \theta \text{ satisfying (6.47).}$$

Then  $y(\theta)$  and  $z(\theta)$  are bijective in (6.47). Hence we regard  $y$  as a function of  $z$  on this domain denoted by the same symbol  $y = y(z)$ . Changing variables, we have

$$(6.51) \quad \left| \int_x^{-1} \left( \frac{1}{f(y, n)} \right)^{1/4} \cos(g(y, n) - \frac{\theta}{2}) dy \right| = \left| \int_\ell^L \frac{G(y(z))}{z^{-1/2}} \frac{\cos z}{z^{1/2}} dz \right|,$$

where  $L = z(x)$ ,  $\ell = z(-1)$ , and  $G(y(z)) = -(\frac{1}{f(y(z), n)})^{1/4} \frac{\partial y}{\partial z}$ .

Recall that  $f(y, n) = n^{2/3} \sin^2 \theta$  and note that by (6.48) and (6.49)

$$-\frac{\partial y}{\partial z} = \frac{2n^{1/6} \sin \theta}{2\sqrt{n+1} \sin^2 \theta - (2\sqrt{n+1})^{-1}}.$$

Then we deduce from this and (6.50) that for  $\theta$  satisfying (6.47)

$$G(y(z)) = \frac{2\sqrt{\sin \theta}}{2\sqrt{n+1} \sin^2 \theta - (2\sqrt{n+1})^{-1}} > 0.$$

Hence we obtain that

$$(6.52) \quad \begin{aligned} \frac{G(y(z))}{z^{-1/2}} &= \frac{2\sqrt{\sin \theta}}{2\sqrt{n+1} \sin^2 \theta - (2\sqrt{n+1})^{-1}} \sqrt{\frac{n+1}{2}(2\theta - \sin 2\theta) - \frac{\theta}{2}} \\ &= \frac{\sqrt{2 \sin \theta}}{2 \sin^2 \theta - (2(n+1))^{-1}} \sqrt{(2\theta - \sin 2\theta) - \frac{\theta}{n+1}}. \end{aligned}$$

By a simple calculation we see that for  $\theta$  satisfying (6.47)

$$(6.53) \quad n^{-1/3} \leq c_{16} \sin \theta, \quad 0 < \frac{G(y(z))}{z^{-1/2}} \leq c_{17}$$

with some constants  $c_{16}$  and  $c_{17}$  independent of  $(\theta, n)$  and  $(z, n)$ , respectively. Furthermore, from (6.50) and (6.52) we can take  $c_{16}$  such that  $G(y(z))/z^{-1/2}$  non-increasing in  $z \in \{z(\theta); n^{-1/3} \leq c_{16} \sin \theta\}$ . Hence (6.45) is derived from (6.51) combined with the second inequality in (6.53).  $\square$

## 6.5 Appendix 5: Proof of Lemma 4.5

In this subsection we complete the proof of Lemma 4.5.

For  $a, b, c \in \mathbb{R}$  we introduce functions  $G_{a,b,c}$  and  $\widehat{G}_{a,b,c}$  on  $\mathbb{R}^2$ , defined as

$$(6.54) \quad G_{a,b,c}(x, y) = \frac{1}{|x|^a |y|^b |x - y|^c},$$

$$(6.55) \quad \widehat{G}_{a,b,c}(x, y) = G_{a,b,c}(x, y) + G_{a,b,c}(y, x).$$

Note that  $G_{a,b,c}(x, y)G_{a',b',c'}(x, y) = G_{a+a', b+b', c+c'}(x, y)$  and that

$$G_{a,b,c}(x, y) \leq G_{a,b,c}(|x|, |y|).$$

**Lemma 6.11.** *Let  $K_{\text{Ai},2}^n$  be the kernel as in (6.21).*

(i) *There exists  $c_{18} > 0$  such that for  $x, y \in [-2n^{2/3}, \infty)$ ,  $n \in \mathbb{N}$ ,*

$$(6.56) \quad |K_{\text{Ai},2}^n(x, y)| \leq c_{18} \widehat{G}_{-1/4,1/4,1}(x, y),$$

$$(6.57) \quad \left| \frac{\partial K_{\text{Ai},2}^n(x, y)}{\partial y} \right| \leq c_{18} \left\{ \widehat{G}_{-1/4,1/4,2}(x, y) + \widehat{G}_{-3/4,1/4,1}(x, y) \right\}.$$

(ii) *For each  $r > 0$ , there exists  $c_{19} > 0$  such that for any  $y \in [-2n^{2/3}, \infty)$ ,  $n \in \mathbb{N}$  with  $|y| > r + 1$ ,*

$$(6.58) \quad \max_{x \in [-r, r]} |K_{\text{Ai},2}^n(x, y)| \leq c_{19} |y|^{-3/4},$$

$$(6.59) \quad \max_{x \in [-r, r]} \left| \frac{\partial K_{\text{Ai},2}^n(x, y)}{\partial y} \right| \leq c_{19} |y|^{-1/4}.$$



(iii) *There exists  $c_{20} > 0$  such that*

$$(6.60) \quad \sup_{y \in (x-1, x+1)} |K_{\text{Ai}, 2}^{\mathbf{n}}(x, y)| \leq c_{20}(|x|^{1/2} + 1), \quad x \in [-2\mathbf{n}^{2/3}, \infty), \quad \mathbf{n} \in \mathbb{N},$$

$$(6.61) \quad \sup_{y \in (x-1, x+1)} \left| \frac{\partial K_{\text{Ai}, 2}^{\mathbf{n}}(x, y)}{\partial y} \right| \leq c_{20}(|x| + 1), \quad x \in [-2\mathbf{n}^{2/3}, \infty), \quad \mathbf{n} \in \mathbb{N},$$

$$(6.62) \quad \sup_{y \in \mathbb{R}} \left| \int_x^y K_{\text{Ai}, 2}^{\mathbf{n}}(u, x) du \right| \leq c_{20}, \quad x \in \mathbb{R}, \quad \mathbf{n} \in \mathbb{N}.$$

*Proof.* We deduce (6.56) from (6.21) combined with (6.33) and (6.34) in Lemma 6.9. We deduce (6.57) from (6.19) in addition to the equations above. We thus obtain (i). The claims in (ii) are derived from (i).

Next, we prove (iii). To obtain (6.60) and (6.61), we use (6.19) and (6.21) in addition to Lemma 6.9 and apply Taylor's theorem. The last claim (6.62) is derived by the same argument as that used to obtain (6.11) and the estimates in Lemma 6.8.  $\square$

**Lemma 6.12.** *Let  $\beta = 1, 4$ . Let  $J_{\beta}^{\mathbf{n}}$  be as in (6.23). Then there exists  $c_{21} > 0$  independent of  $\mathbf{n}$ , satisfying for  $x, y \in [-2\mathbf{n}^{2/3}, \infty)$*

$$(6.63) \quad |J_{\beta}^{\mathbf{n}}(x, y)| \leq c_{21} \left\{ \widehat{G}_{-1/4, 1/4, 1}(x, y) + |x|^{-1/4} \right\},$$

$$(6.64) \quad \left| \frac{\partial J_{\beta}^{\mathbf{n}}(x, y)}{\partial y} \right| \leq c_{21} \{ \widehat{G}_{-1/4, 1/4, 2}(x, y) + \widehat{G}_{-3/4, 1/4, 1}(x, y) \},$$

$$(6.65) \quad \left| \int_x^y J_{\beta}^{\mathbf{n}}(u, x) du \right| \leq c_{21}.$$

*Proof.* We deduce Lemma 6.12 from (6.23), Lemma 6.9, Lemma 6.10, and Lemma 6.11.  $\square$

**Lemma 6.13.** *Let  $\beta = 1, 4$ . Let  $L_{\beta}^{\mathbf{n}}$  be as in (4.15). Then there exists  $c_{22} > 0$  independent of  $\mathbf{n}$  such that for  $x, y \in [-2\mathbf{n}^{2/3}, \infty)$*

$$(6.66) \quad |L_{\beta}^{\mathbf{n}}(x, y)| \leq c_{22} \left\{ \widehat{G}_{-1/4, 1/2, 1}(x, y) + \widehat{G}_{0, 1/4, 1}(x, y) + \widehat{G}_{1/4, 1/4, 0}(x, y) \right. \\ \left. + \widehat{G}_{-3/4, 1/4, 1}(x, y) + \widehat{G}_{-1/2, 1/2, 2}(x, y) + \widehat{G}_{0, 0, 2}(x, y) \right\}.$$

*Proof.* We deduce Lemma 6.13 from Lemma 6.7 and Lemma 6.12.  $\square$

**Lemma 6.14.** *Let  $\beta = 1, 2, 4$ . Then for any  $x \in \mathbb{R}$ , there exists  $c_{23} > 0$  such that for any  $y \in [-2\mathbf{n}^{2/3}, \infty)$ ,  $\mathbf{n} \in \mathbb{N}$  with  $|y| > |x| + 1$ ,*

$$(6.67) \quad |\rho_{\text{Ai}, \beta, x}^{\mathbf{n}, 1}(y) - \rho_{\text{Ai}, \beta}^{\mathbf{n}, 1}(y)| \leq c_{23} \{ |y|^{-3/2} + \mathbf{1}(\beta \neq 2) |y|^{-1/4} \}.$$

*Proof.* From (4.1)–(4.3) and (4.15) we see that

$$(6.68) \quad |\rho_{\text{Ai}, \beta, x}^{\text{n},1}(y) - \rho_{\text{Ai}, \beta}^{\text{n},1}(y)| = \frac{|L_{\beta}^{\text{n}}(x, y)|}{\rho_{\text{Ai}, \beta}^{\text{n},1}(x)}.$$

Let  $\beta = 2$ , Then (6.67) is derived from (6.68) combined with (6.58) and (4.15). Let  $\beta = 1, 4$ . Then we obtain (6.67) from (6.68) and Lemma 6.13 combined with (6.54) and (6.55) immediately.  $\square$

We use the following lemma to prove (4.16).

**Lemma 6.15.** *Suppose that  $g$  is a non-negative function on  $\mathbb{R}^2$  satisfying for some  $a_i, b_i, c_i, \nu_i, c_{24} \in \mathbb{R}$  ( $i = 1, \dots, m$ ) such that  $0 < a_i + b_i + c_i$ ,  $0 \leq \nu_i$ , and  $0 < c_{24}$ ,*

$$(6.69) \quad g(u, v) \leq c_{24} \sum_{i=1}^m \widehat{G}_{a_i, b_i, c_i}(|u|, |v|) \quad \text{for all } u, v \in \mathbb{R},$$

$$(6.70) \quad \sup_{\{v; |u-v| \leq 1\}} g(u, v) \leq c_{24} \sum_{i=1}^m (1 + |u|^{\nu_i}) \quad \text{for all } u \in \mathbb{R}.$$

Assume that there exist constants  $\gamma_i$  and  $\kappa_i$  such that

$$(6.71) \quad \max\{0, \nu_i - 1\} < \gamma_i, \quad 0 < 1 + a_i + b_i - \kappa_i + (\kappa_i + c_i - 1)\gamma_i.$$

Then,

$$(6.72) \quad \lim_{s \rightarrow \infty} \int_{s \leq |u|, s \leq |v|} \frac{g(u, v)}{|uv|} du dv = 0.$$

*Proof.* Without loss of generality, we can assume  $m = 1$ . Hence we write  $a = a_1$ ,  $b = b_1$ ,  $c = c_1$ , and so on. Since the assumptions in (6.69), (6.70) and (6.71) are essentially symmetric in  $u$  and  $v$ , and depends only on the absolute values  $|u|$  and  $|v|$ , it is sufficient for (6.72) to prove

$$(6.73) \quad \lim_{s \rightarrow \infty} \int_s^\infty du \int_u^\infty dv \frac{g(u, v)}{uv} = 0.$$

Dividing the set  $[u, \infty)$  into  $[u, u + u^{-\gamma})$  and  $[u + u^{-\gamma}, \infty)$  and changing variables such that  $w = v - u$ , we obtain that

$$(6.74) \quad \int_s^\infty du \int_u^\infty dv \frac{g(u, v)}{uv} = \int_s^\infty du \left\{ \int_0^{u^{-\gamma}} dw + \int_{u^{-\gamma}}^\infty dw \right\} \frac{g(u, u + w)}{u(u + w)}.$$

By (6.71) we take  $\gamma > \max\{0, \nu - 1\}$ . Then from (6.70) we see that

$$\begin{aligned}
(6.75) \quad \int_s^\infty du \int_0^{u^{-\gamma}} \frac{g(u, u+w)}{u(u+w)} dw &\leq c_{24} \int_s^\infty du \int_0^{u^{-\gamma}} \frac{1+u^\nu}{u(u+w)} dw \\
&\leq c_{24} \int_s^\infty du \left\{ \frac{1+u^\nu}{u^2} \right\} \int_0^{u^{-\gamma}} dw \\
&= O(s^{-1+\nu-\gamma}).
\end{aligned}$$

From (6.69) and (6.71) we deduce that

$$\begin{aligned}
(6.76) \quad \int_s^\infty du \int_{u^{-\gamma}}^\infty \frac{g(u, u+w)}{u(u+w)} dw \\
&\leq c_{24} \int_s^\infty du \int_{u^{-\gamma}}^\infty \frac{1}{u(u+w)} \frac{1}{u^a(u+w)^b} \frac{1}{w^c} dw \\
&\leq c_{24} \int_s^\infty du \int_{u^{-\gamma}}^\infty \frac{1}{u^{2+a+b-\kappa}} \frac{1}{w^{\kappa+c}} dw \\
&= O(s^{-(1+a+b-\kappa+(\kappa+c-1)\gamma)}).
\end{aligned}$$

Hence (6.73) follows from (6.74), (6.75), and (6.76) combined with (6.71).  $\square$

We are now ready to prove Lemma 4.5.

*Proof of Lemma 4.5.* Suppose  $k = 1$ . Then we see (4.16) from (4.9) easily.

Let  $\psi_n^{[m]} = d^m \psi_n / dx^m$  for  $m = 0, 1, 2$  and  $x^* = -(4n^{2/3} + x)$ . Then from (6.17)

$$(6.77) \quad |\psi_n^{[m]}(x)| = |\psi_n^{[m]}(x^*)| \quad \text{for } m = 0, 1, 2.$$

We set  $\Lambda_n = [-2n^{2/3}, \infty)$  and for  $x \in \mathbb{R}$ . Let

$$\tilde{K}_{\text{Ai}, \beta}^n(x, y) = K_{\text{Ai}, \beta}^n(x, y) \mathbf{1}_{\Lambda_n}(x) \mathbf{1}_{\Lambda_n}(y).$$

Define  $\tilde{I}_{\beta, k}^n$  and  $\tilde{L}_\beta^n$  from  $I_{\beta, k}^n$  and  $L_\beta^n$  by substituting  $K_{\text{Ai}, \beta}^n$  into  $\tilde{K}_{\text{Ai}, \beta}^n$ , respectively. Then (4.16) for  $2 \leq k \leq 6$  follows from

$$(6.78) \quad \lim_{s \rightarrow \infty} \sup_{2 \leq n \in \mathbb{N}} \sup_{|x| \leq r} \tilde{I}_{\beta, k}^n(x, s) = 0 \quad \text{for all } 2 \leq k \leq 6.$$

In fact, we can easily see from (6.77) that  $\tilde{I}_{\beta, k}^n$  is a main part of  $I_{\beta, k}^n$ .

Suppose  $k = 2, 3$ . From (6.58), and Lemma 6.13 we deduce that there exists  $c_{25} > 0$  such that

$$(6.79) \quad \sup_{x \in [-r, r]} \sup_{n \in \mathbb{N}} |\tilde{L}_\beta^n(x, y)| \leq c_{25} |y|^{-1/4}, \quad |y| > r + 1.$$

Then, we deduce (6.78) with  $k = 2, 3$  from (6.79) easily.

Suppose  $k = 4$ . If  $\beta = 2$ , then we deduce (6.78) from Lemma 6.15 with (6.56), (6.58), and (6.60). Suppose  $\beta = 1$ . Then from (6.23), (6.29), and Lemma 6.11 (iii) there exists  $c_{26} > 0$  independent of  $n$  such that

$$(6.80) \quad \sup_{v \in [u-1, u+1]} \sup_{n \in \mathbb{N}} |\tilde{L}_\beta^n(u, v)| \leq c_{26}(1 + |u|), \quad u \in \mathbb{R}.$$

From (6.66) and (6.80), the assumptions in Lemma 6.15 are fulfilled, which yields

$$(6.81) \quad \lim_{s \rightarrow \infty} \int_{s \leq |u|, s \leq |v|} \frac{1}{|uv|} \sup_{n \in \mathbb{N}} |\tilde{L}_\beta^n(u, v)| du dv = 0.$$

Eq. (6.78) follows from (6.81) easily. The proof for  $\beta = 4$  is similar to that of  $\beta = 1$ . Hence we omit it.

We suppose  $k = 5$ . If  $\beta = 2$ , then we obtain (6.78) from Lemma 6.11 and Lemma 6.15. Next suppose  $\beta = 1$ . Since  $[K_{\text{Ai},1}^n(u, x)K_{\text{Ai},1}^n(x, v)K_{\text{Ai},1}^n(v, u)]^{[0]}$  equals

$$\begin{aligned} & J_1^n(u, x)J_1^n(x, v)J_1^n(v, u) - J_1^n(u, x)\frac{\partial J_1^n(x, v)}{\partial v} \left\{ \int_u^v J_1^n(t, u)dt - \frac{\text{sign}(v - u)}{2} \right\} \\ & - \frac{\partial J_1^n(u, x)}{\partial x} \left\{ \int_v^x J_1^n(t, v)dt - \frac{\text{sign}(x - v)}{2} \right\} J_1^n(v, u) \\ & - \frac{\partial J_1^n(u, x)}{\partial x} J_1^n(x, v) \left\{ \int_u^v J_1^n(t, u)dt - \frac{\text{sign}(v - u)}{2} \right\}, \end{aligned}$$

we can obtain (6.78) from Lemma 6.12 by the same argument used in the above. Finally, the case of  $\beta = 4$  follows from the same argument as used in the case of  $\beta = 1$ .

The case  $k = 6$  can be proved similarly as the case of  $k = 5$ . □

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